GEOMETRIC MODULAR ACTION

By

MARTIN FLORIG

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1999

ACKNOWLEDGMENTS

First and foremost, I want to thank my adviser, Professor Stephen J. Summers, for his constant guidance and encouragement. Working with him is always a great pleasure.

I would also like to thank Professor P. Ehrlich, Professor G. Emch, Professor J. Klauder, Professor K. Muttalib, and Professor P. Robinson for their service on my supervisory committee.

I am especially grateful to my wife, Nicole, and to my parents.

TABLE OF CONTENTS

page

ACKNOWLEDGMENTS ii

ABSTRACT iv

CHAPTERS

1 INTRODUCTION 1

2 HALF-SIDED TRANSLATIONS 8

- 2.1 Borchers' Theorem 8
- 2.2 Half-Sided Modular Inclusions 13
- 2.3 Intermediate Algebras 14
- 2.4 Additivity 20
- 2.5 Conformal Quantum Field Theory 22

3 GEOMETRIC MODULAR ACTION ON MINKOWSKI SPACE 28

- 3.1 Bijections of Wedges 30
- 3.2 Which Transformation Groups Can Appear? 42
  - 3.2.1 Two-Dimensional Minkowski Space 43
  - 3.2.2 Four-Dimensional Minkowski Space 51
- 3.3 Transitive Action 68
- 3.4 Geometric Action of Even Products of Modular Conjugations . . 70
- 3.5 Construction of the Poincaré Group 77
  - 3.5.1 The Translation Group 78
  - 3.5.2 The Lorentz Group 81
  - 3.5.3 The Poincaré Group 94
- 3.6 Geometric Action of Modular Groups 98
  - 3.6.1 Modular Groups 100
  - 3.6.2 Modular Conjugations 105

4 GEOMETRIC MODULAR ACTION ON DE SITTER SPACE 108

- 4.1 Bijections of Wedges 110
- 4.2 Construction of the de Sitter Group 122

REFERENCES 130

BIOGRAPHICAL SKETCH 133

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

GEOMETRIC MODULAR ACTION

By

Martin Florig

May 1999

Chairman: Stephen J. Summers
Major Department: Mathematics

We apply a selection criterion for physically interesting states on nets of C^*-algebras over general spacetimes, which allows the determination of spacetime symmetry groups and their corresponding representations from the algebras of the net and a suitable state. In particular, we examine nets over Minkowski space and de Sitter space.

We consider half-sided modular inclusions. Simplified proofs of two well-known theorems, which are due to Borchers and Wiesbrock, are given. We show that a half-sided modular inclusion can never be split and we apply a theorem which can be used in algebraic quantum field theory in order to derive additivity for von Neumann algebras associated with regions with nontrivial intersection.

CHAPTER 1
INTRODUCTION

In algebraic quantum field theory the existence of spacetime symmetries is often assumed from the outset. However, the physical information should already be obtainable from the net of local algebras, i.e., from the map $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$, which assigns to every spacetime region $\mathcal{O}$ the corresponding C^*-algebra $\mathcal{A}(\mathcal{O})$ of in $\mathcal{O}$ measurable observables.

Bisognano and Wichmann [4, 5] showed for finite-component quantum field theories satisfying Wightman's axioms that the modular objects associated with the vacuum and wedge algebras have a geometric interpretation. The modular objects only depend on the corresponding algebra and state; i.e., it is possible to derive geometrical and dynamical information as well as representations of symmetry groups merely by using a net of local algebras and a suitable state.

Numerous other examples are known where the modular objects associated with certain algebras and the vacuum act geometrically. In conformal quantum field theory this holds for double cone algebras [28] and for light cone algebras [15]. In quantum field theories on the circle the modular objects associated with interval algebras have a geometric interpretation [41]. In de Sitter space this holds again for wedge algebras [9, 12].

In the literature [4, 5, 9, 12, 15, 28, 41] there are many examples of half-sided modular inclusions (see Definition 2.1.2), which we consider in Chapter 2. We will first prove a theorem which is due to Borchers [7]. Our proof can also be applied to derive Wiesbrock's results [40, 42]. We only use elementary theorems from complex analysis, thereby significantly simplifying the original proofs. Furthermore, we will show that pairs of von Neumann algebras of a half-sided modular inclusion can never be split. Then we give some applications of a theorem which can be used in algebraic quantum field theory to derive additivity for von Neumann algebras associated with regions with nontrivial intersection.

An algebraic characterization of vacuum states on nets of C^*-algebras over Minkowski space was given by Buchholz and Summers [17]. It allows the reconstruction of spacetime translations by means of the modular conjugations which are associated with these states. Furthermore, the following selection criterion for physically interesting states of quantum field theories on general spacetimes was proposed. It makes possible the determination of spacetime symmetry groups and their corresponding representations, as well as the characterization of vacuum states by means of algebraic conditions.

Condition of Geometric Modular Action (CGMA): *Let $\mathcal{W}$ be a suitable collection of regions of the spacetime $(\mathcal{M}, g)$, $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ a net of C^*-algebras, which lie in a C^*-algebra $\mathcal{A}$, and ω a state on $\mathcal{A}$ with GNS representation $(\mathcal{H}, \pi, \Omega)$. Let Ω be cyclic and separating for $\mathcal{R}(W) = \pi(\mathcal{A}(W))''$, i.e., the modular conjugations J_W of $(\mathcal{R}(W), \Omega)$ are defined for $W \in \mathcal{W}$.*

The pair $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}}, \Omega)$ satisfies the condition of geometric modular action if the adjoint action of the modular conjugations J_W leaves the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant, i.e., if, for all $W, W_1 \in \mathcal{W}$, there is a $W_2 \in \mathcal{W}$ such that

$$J_W \mathcal{R}(W_1) J_W \; = \; \mathcal{R}(W_2).$$

It is remarkable that this condition can be formulated on arbitrary spacetimes, for we do not assume any specific action of the modular conjugations.

The CGMA allows the derivation of representations of symmetry groups merely from a net $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ and a suitable state, since the modular conjugations J_W are determined by the pairs $(\mathcal{R}(W),\Omega)$. In Chapter 3 and Chapter 4 we will study the CGMA. We will consider nets of C*-algebras over Minkowski space and de Sitter space. Under the assumption of the CGMA and additional purely algebraic conditions we will show that the modular conjugations of wedge algebras induce a strongly continuous representation of the proper Poincaré group, resp. of the proper Lorentz group. Many of these results already appeared [18] (see also [36]).

Furthermore, we will consider modified versions of the CGMA. We will only assume that even products of modular conjugations or that the elements of the modular groups of $(\mathcal{R}(W),\Omega)$ have a geometric action.

Algebraic characterizations of vacuum states over Minkowski space were proposed by Doplicher [20] and Kraus [30]. However, these criteria cannot be applied to general spacetimes. Other selection criteria, such as the Hadamard condition [29] or the microlocal spectrum condition [35], are valid for a whole folium, i.e., they do not allow a decision about which states are fundamental. Furthermore, the Hadamard condition only restricts the two-point function (therefore, it is of particular interest for quasifree states).

Since the CGMA does not assume a specific action of the modular conjugations, it can be applied to nets on arbitrary spacetimes. This generality implies, however, that the symmetry groups which can occur have to be determined. We want to explain how we have to proceed if we want to apply the CGMA to nets of C^*-algebras over a spacetime $(\mathcal{M},g)$.

1) How should we choose the regions in $\mathcal{W}$?

There are some natural requirements on the set $\mathcal{W}$ [18]. In the examples which were considered so far, wedges are used to define the elements of $\mathcal{W}$. In Minkowski space the causal completions of inextendible timelike curves give such regions if the completions are not equal to the whole space. It would be interesting to determine for which spacetimes we can define suitable regions in this way. However, we can only give a justification of a certain choice of $\mathcal{W}$ by showing that a large class of quantum field theories then satisfies the CGMA. Therefore, we should prove Bisognano-Wichmann type theorems as in [4, 5, 9, 12].

The adjoint action of the modular conjugations permutes the set of von Neumann algebras $\{\mathcal{R}(W) \mid W \in \mathcal{W}\}$. Since we only consider nets for which the map $\mathcal{W} \ni W \mapsto \mathcal{R}(W)$ is bijective, we can define permutations τ_W of $\mathcal{W}$ by

$$J_W \mathcal{R}(W_0) J_W \;=\; \mathcal{R}(\tau_W(W_0)) \quad \forall W, W_0 \in \mathcal{W}.$$

In the following we denote by $\mathcal{T}$ the group generated by the permutations $\{\tau_W \mid W \in \mathcal{W}\}$.

2) Are the transformations in $\mathcal{T}$ induced by point transformations on the spacetime $(\mathcal{M}, g)$?

In addition to the CGMA, we will assume further purely algebraic conditions which can also be formulated on arbitrary spacetimes. In the examples we consider below we can show then that $\cap_{k=1}^{n} \overline{W_k}$, $W_1, \ldots, W_n \in \mathcal{W}$, is a singleton set iff $\cap_{k=1}^{n} \overline{\tau_W(W_k)}$ is a singleton set. Finally we can define point transformations implementing τ_W. However, it is not desirable that these arguments can be generalized to arbitrary spacetimes $(\mathcal{M}, g)$. Since the CGMA does not assume a specific action of the modular conjugations, it is possible that they do not act as point transformations. This is an advantage if we consider spacetimes with a small or trivial symmetry group. Even in such cases the CGMA could characterize physically interesting states.

3) Which subgroups $\mathcal{T}$ of the symmetric group on $\mathcal{W}$ can appear, and does the abstract group $\mathcal{T}$ already determine the geometric action of the modular conjugations on $\mathcal{W}$?

Alexandrov has shown that bijections on Minkowski space preserving the causal structure act as elements of the extended Poincaré group [2]. Similar results are available for many other spacetimes; i.e., in these cases it is possible to show that the modular conjugations act on $\mathcal{W}$ as symmetries if their action can be implemented by point transformations.

Even if the transformations in $\mathcal{T}$ do not act as point transformations, $\mathcal{T}$ has to fulfill the following conditions. $\mathcal{T}$ is generated by involutions τ_W, which satisfy $\sigma\tau_W\sigma^{-1} = \tau_{\sigma W}$, for $\sigma \in \mathcal{T}$ [18, Lemma 2.1]. In particular, τ_W commutes with the elements of $\mathcal{T}$ leaving W invariant. Since we assume isotony, we can further derive

$$\tau_{W_1}\tau_{W_2}(W_2) \subset W_1 \quad \text{for} \quad W_1 \subset W_2$$

(see (3.2.1.1)). All these relations demonstrate that the permutations τ_W and the group $\mathcal{T}$ are subject to strong conditions.

Since τ_W commutes with the elements of $\mathcal{T}$ which leave W invariant, we expect that in many cases the geometric action of the modular conjugations is already uniquely determined by the abstract group $\mathcal{T}$ if $\mathcal{T}$ is "large" enough. A natural way to formulate this mathematically is the requirement that $\mathcal{T}$ acts transitively on the net. By explicit calculations we will show under this assumption for the examples we consider below that to every element of the identity component of the symmetry group of the spacetime there is a product of modular conjugations with this geometric action.

4) If $\mathcal{T}$ acts on $\mathcal{W}$ as a continuous group $\mathcal{G}$ of point transformations, when do the modular conjugations induce a continuous unitary representation of $\mathcal{G}$?

In order to solve this problem we can use Moore's cohomology theory (see [34] or [18]). We denote by $\mathcal{J}$ the group consisting of products of modular conjugations and by $\varphi : \mathcal{J} \mapsto \mathcal{G}$ the homomorphism assigning to each modular conjugation J_W the corresponding element of $\mathcal{G}$ which implements the action of J_W, resp. τ_W, on $\mathcal{W}$. It is easy to show that $(\mathcal{J}, \varphi)$ is a central extension of $\mathcal{G}$, i.e. the kernel of φ lies in the center of $\mathcal{J}$ (see the introduction to Chapter 3) and $\varphi : \mathcal{J} \mapsto \mathcal{G}$ is an epimorphism.

Moore considered central extensions (E, π) of separable, locally compact topological groups G, where π is open and continuous. If G is perfect, then there exists a universal covering group $(\tilde{G}, \nu)$ of G; i.e., to every central extension (E, π) of G by a Polish group $\ker \pi$ (if we consider groups of unitary operators on separable Hilbert spaces, then $\ker \pi$ is necessarily Polish) there is a homomorphism $\chi : \tilde{G} \mapsto E$ with $\pi \circ \chi = \nu$. For semi-simple Lie groups G, $\tilde{G}$ is identical to the topologically defined universal covering group of G.

We want to apply Moore's cohomology theory to a suitable subgroup of $\mathcal{G}$. This is at least possible if $\mathcal{T}$ is "large" enough, since it can easily be shown that the group $\mathcal{J}_e$ consisting of the even products of modular conjugations is perfect if $\mathcal{J}_e'$ acts transitively on $\mathcal{W}$ (see below). In such cases $\mathcal{G}'$ is perfect and we can apply Moore's theory to the central extension $\mathcal{J}_e$ of the group $\mathcal{G}'$. It only remains to show that the homomorphism $\varphi|_{\mathcal{J}_e} : \mathcal{J}_e \mapsto \mathcal{G}'$ is continuous and open (see below). Finally we obtain a continuous representation of the universal covering group of $\mathcal{G}'$, which we can extend to a representation of a covering group of $\mathcal{G}$.

5) How can we characterize physical stability?

If a one-parameter subgroup of $\mathcal{G}$ can be interpreted as time development, then we mean by physical stability that the spectrum of the generator of the corresponding continuous unitary representation, interpreted as the total energy, is bounded from below. However, a spacetime does not admit in general a timelike Killing vector field and therefore no symmetry group can be interpreted as time development. In [18] we consider therefore an algebraic condition, which can be applied to arbitrary spacetimes. We call modular stability the condition that the modular groups of the pairs $(\mathcal{R}(W), \Omega)$ are contained in $\mathcal{J}$. This condition implies for nets on Minkowski space the relativistic spectrum condition. Furthermore, modular stability makes the application of Moore's cohomology theory easier, since the continuity and openness of $\varphi|_{\mathcal{J}_e}$ already follows if $\mathcal{G}$ is a Lie group and if $\mathcal{J}_e$ is generated by strongly continuous one-parameter subgroups (see below or [13]).

CHAPTER 2
HALF-SIDED TRANSLATIONS

We will give a simplified proof of the following important theorem, which is due to Borchers. Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω and let $U(t)$, $t \in \mathbf{R}$, be a group with positive generator, which leaves Ω fixed and induces for positive arguments endomorphisms of $\mathcal{M}$. The modular group of $(\mathcal{M}, \Omega)$ acts then as dilatations on this group. Our proof can also be applied to half-sided modular inclusions.

Given a nontrivial half-sided modular inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$ we show in 2.3 that the pair $(\mathcal{N}, \mathcal{M})$ is not split.

In 2.4 and 2.5 we give some applications of a theorem which can be used in various cases to derive additivity for local algebras associated to regions with non-trivial intersection.

2.1 Borchers' Theorem

Bisognano and Wichmann showed that for finite-component quantum field theories which satisfy the Wightman axioms the modular groups of wedge algebras act as the Lorentz boosts leaving the respective wedges invariant [4, 5]. If modular covariance holds, i.e., if the modular groups act in this manner, it is possible to derive the existence of a strongly continuous representation of the Poincaré group by using half-sided modular inclusions [8], which are closely related to Borchers' theorem (see below).

Even in settings where modular covariance does not hold, Borchers' theorem provides necessary and sufficient conditions that a strongly continuous unitary representation of the translation group satisfies the spectrum condition (the necessity was proven in [17] and, under slightly more restrictive conditions, in [39]). Furthermore, it shows that such a representation, if it exists, is uniquely defined by the modular objects of the wedge algebras (see equation (2.1.4), resp. [17] if modular conjugations are to be used).

Borchers' theorem, resp. half-sided modular inclusions, has been crucially employed in many other publications [10, 18, 25, 41, 43], to name only a few examples. It has even been applied to quantum fields defined on spacetimes different from Minkowski space [37]. But the original proof of this deep result is quite complicated. In the next section, we will give a proof of the theorem by using, besides modular theory, only elementary theorems from complex analysis, thereby significantly simplifying Borchers' original proof. We state Borchers' theorem, which was first formulated in [7].

Theorem 2.1.1: *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with cyclic and separating vector $\Omega \in \mathcal{H}$. Denote by Δ the modular operator and by J the modular conjugation with respect to $(\mathcal{M}, \Omega)$. Let $U(a)$, $a \in \mathbf{R}$, be a group with positive generator acting on $\mathcal{H}$, leaving Ω fixed, and satisfying*

$$U(a)\mathcal{M}U(-a) \;\subset\; \mathcal{M} \tag{2.1.1}$$

for $a \geq 0$. These conditions imply

$$\Delta^{it}U(a)\Delta^{-it} \;=\; U(e^{-2\pi t}a) \quad \forall a, t \in \mathbf{R} \tag{2.1.2}$$

and

$$JU(a)J \;=\; U(a)^{*} \quad \forall a \in \mathbf{R}. \tag{2.1.3}$$

Motivated by the appearance of this result, Wiesbrock defined and studied half-sided modular inclusions [41, 43].

Definition 2.1.2: Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras on a Hilbert space $\mathcal{H}$ and Ω a common cyclic and separating vector. Denote by $\Delta_{\mathcal{M}}$, $\Delta_{\mathcal{N}}$, resp. $J_{\mathcal{M}}$, $J_{\mathcal{N}}$, the modular operators, resp. modular conjugations. Assume $\Delta_{\mathcal{M}}^{-it}\mathcal{N}\Delta_{\mathcal{M}}^{it} \subset \mathcal{N}$ for $\mp t \geq 0$. Under these circumstances, the pair $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is called a $\pm$half-sided modular inclusion.

Theorem 2.1.1 entails, given the situation described by its hypothesis, that $(U(a)\mathcal{M}U(-a) \subset \mathcal{M}, \Omega)$, $a \geq 0$, are -half-sided modular inclusions ($t \geq 0$):

$$\begin{aligned} & \Delta_{\mathcal{M}}^{-it}(U(a)\mathcal{M}U(-a))\Delta_{\mathcal{M}}^{it} \\ = \quad & U(e^{2\pi t}a)\Delta_{\mathcal{M}}^{-it}\mathcal{M}\Delta_{\mathcal{M}}^{it}U(-e^{2\pi t}a) \\ = \quad & U(a)(U((e^{2\pi t}-1)a)\mathcal{M}U(-(e^{2\pi t}-1)a))U(-a) \\ \subset \quad & U(a)\mathcal{M}U(-a) \end{aligned}$$

On the other hand, given a -half-sided modular inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$ we can define a group $U(t)$, $t \in \mathbf{R}$, by

$$U(1 - e^{-2\pi t}) = \Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it} \tag{2.1.4}$$

such that $\mathcal{M}$, resp. $\mathcal{N}$, $U(t)$, and Ω satisfy the conditions of Borchers' theorem [41, 43]. A generalized version of Theorem 2.1.1 (see Theorem 2.2.1 below) was used to prove the latter assertion. These remarks imply that half-sided modular inclusions and the assumptions of Borchers' theorem are only different descriptions for the same situation. Hence, our arguments can be used in both settings, as we shall see.

If we assume $a \leq 0$ in (2.1.1) and if we consider +half-sided modular inclusions, then we can derive relations as in Theorem 2.1.1 and Theorem 2.2.1, by making the obvious modifications in the proofs below [7, 40].

We will simplify the proof of Borchers' theorem by using the elementary theory of analytic functions in one variable.

Proof of Theorem 2.1.1: Choose arbitrary but fixed $M \in \mathcal{M}$, $M' \in \mathcal{M}'$. We obtain the adjoint equations of (2.1.2) and (2.1.3) if we replace a by $-a$. Hence, we can assume $a \geq 0$. We can define a bounded and continuous function on $S = \{z \in \mathbf{C} \,|\, 0 \leq \operatorname{Im} z \leq \frac{1}{2}\}$ by (the generator of $U(t)$ is positive and $\operatorname{Im} e^{2\pi z} a \geq 0$ for $z \in S$)

$$f(z) \;=\; (\Delta^{-i\bar{z}} M'\Omega, U(e^{2\pi z}a)\Delta^{-iz} M\Omega),$$

which is analytic in the interior of S. We will show that $f(z)$ can be extended to an entire function, which is bounded and therefore constant. Set $V(t) = JU(-t)J$, $t \in \mathbf{R}$. The group $V(t)$ satisfies (2.1.1) too because of $J\mathcal{M}J = \mathcal{M}'$ and

$$[U(-t)M_1'U(t), M_1] \;=\; U(-t)[M_1', U(t)M_1U(-t)]U(t) \;=\; 0$$

for $t \geq 0$, $M_1 \in \mathcal{M}$, $M_1' \in \mathcal{M}'$, which implies $U(-t)\mathcal{M}'U(t) \;\subset\; \mathcal{M}'$ for $t \geq 0$. $V(e^{2\pi t}a)\Delta^{-it}M^*\Delta^{it}V(e^{2\pi t}a)^* \in \mathcal{M}$ and $\Delta^{it}V(e^{2\pi t}a)^*\Omega = \Omega$ entail

$$\begin{aligned}
f(t+\frac{i}{2}) \;&=\; (\Delta^{-\frac{1}{2}}\Delta^{-it}M'\Omega, U(e^{2\pi t + i\pi}a)\Delta^{-it}\Delta^{\frac{1}{2}}M\Omega) \\
&=\; (\Delta^{-\frac{1}{2}}\Delta^{-it}M'\Omega, U(-e^{2\pi t}a)J\Delta^{-it}M^*\Omega) \\
&=\; (\Delta^{-\frac{1}{2}}\Delta^{-it}M'\Omega, J(V(e^{2\pi t}a)\Delta^{-it}M^*\Delta^{it}V(e^{2\pi t}a)^*)\Omega) \\
&=\; (\Delta^{-it}M'\Omega, (J\Delta^{\frac{1}{2}})(V(e^{2\pi t}a)\Delta^{-it}M^*\Delta^{it}V(e^{2\pi t}a)^*)\Omega) \\
&=\; (\Delta^{-it}M'\Omega, (V(e^{2\pi t}a)\Delta^{-it}M^*\Delta^{it}V(e^{2\pi t}a)^*)^*\Omega) \\
&=\; (\Delta^{-it}M'\Omega, V(e^{2\pi t}a)\Delta^{-it}M\Omega) \qquad (2.1.5)
\end{aligned}$$

for $t \in \mathbf{R}$. The group $V(t)$ possesses a positive generator, leaves Ω fixed, and fulfills (2.1.1) too. As above, we can show that there is a bounded continuous function $g(z)$ defined on S and analytic in the interior such that

$$g(z) \;=\; (\Delta^{-i\bar{z}}M'\Omega, V(e^{2\pi z}a)\Delta^{-iz}M\Omega).$$

(2.1.5) and the corresponding equation for $g(z)$ imply

$$f(t+\frac{i}{2}) \;=\; g(t) \quad \text{for} \quad t \in \mathbf{R} \tag{2.1.6}$$

and

$$\begin{aligned} g(t+\frac{i}{2}) \;&=\; (\Delta^{-it}M'\Omega, (JV(-e^{2\pi t}a)J)\Delta^{-it}M\Omega) \\ &=\; (\Delta^{-it}M'\Omega, U(e^{2\pi t}a)\Delta^{-it}M\Omega) \\ &=\; f(t). \end{aligned} \tag{2.1.7}$$

Define now $h(z)$ by $h(z) = f(z-in)$ for $n \leq \operatorname{Im} z \leq n+\frac{1}{2}$ and $h(z) = g(z-i(n+\frac{1}{2}))$ for $n+\frac{1}{2} \leq \operatorname{Im} z \leq n+1$ ($n \in \mathbf{Z}$). Then $h(z)$ is bounded, well-defined, and continuous on $\mathbf{C}$ ((2.1.6), (2.1.7)). Furthermore, it is analytic on $\cup_{k\in\mathbf{Z}}\{z \mid \frac{k}{2} < \operatorname{Im} z < \frac{k+1}{2}\}$. We can conclude that $h(z)$ is entire (use the reflection principle) and therefore constant. $h(t) = h(0)$ ($t \in \mathbf{R}$) entails then ($\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in $\mathcal{H}$)

$$\Delta^{it}U(e^{2\pi t}a)\Delta^{-it} \;=\; \Delta^{i0}U(e^{2\pi 0}a)\Delta^{-i0} \;=\; U(a).$$

$h(0) = h(i/2)$ proves

$$U(a) \;=\; V(a) \;=\; JU(-a)J.$$

2.2 Half-Sided Modular Inclusions

Our proof of Borchers' theorem can easily be modified to show the following (see [40, 42]).

Theorem 2.2.1: *Let* $U(a)$, $a \in \mathbf{R}$, *be a family of unitary operators leaving* Ω *fixed* ($\mathcal{M}$, $\mathcal{H}$, Ω, Δ, *and* J *as in Theorem 2.1.1) and satisfying*

- *(i)* $U(a)$ *can be extended to a bounded and continuous function on* S, *which is analytic in the interior of* S.
- *(ii)* $U(a)\mathcal{M}U(a)^* \subset \mathcal{M}\ \forall a \in \mathbf{R}$
- *(iii)* $U(a+i/2)\mathcal{M}'U(a+i/2)^* \subset \mathcal{M}'\ \forall a \in \mathbf{R}$

These conditions imply

$$\Delta^{it}U(a)\Delta^{-it} = U(a-t) \quad \forall a, t \in \mathbf{R} \tag{2.2.1}$$

and

$$JU(a)J = U(a+i/2) \quad \forall a \in \mathbf{R}. \tag{2.2.2}$$

The conclusion of the theorem still holds if $U(z)$ *is only defined on* $S \backslash S_0$ *where* S_0 *is a finite subset of* S.

The last remark follows easily from the fact that isolated singularities of bounded analytic functions are removable. In [40] and [42], the preceding theorem was used to prove [40, Theorem 3], according to which we can define a group $U(t)$, $t \in \mathbf{R}$, by

$$U(e^{2\pi s}-1) = \Delta_{\mathcal{M}}^{-is}\Delta_{\mathcal{N}}^{is}, \quad s \in \mathbf{R},$$

such that $\mathcal{M}$, resp. $\mathcal{N}$, $U(t)$, and Ω satisfy the conditions of Borchers' theorem. It was shown that $U(e^{2\pi z})$, $z \in S\backslash\{i/2\}$, and $\mathcal{N}$ fulfill (i)-(iii) of Theorem 2.2.1. Then, Theorem 3 in [40] can be readily derived from (2.2.1) and (2.2.2).

2.3 Intermediate Algebras

Given a half-sided modular inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$ we will show that the split property cannot hold for the pair $(\mathcal{N}, \mathcal{M})$, i.e. that there is not an intermediate type I factor $\mathcal{L}$ $(\mathcal{N} \subset \mathcal{L} \subset \mathcal{M})$. Under certain additional conditions this assertion was already proven in [40].

Half-sided modular inclusions play an important role in algebraic quantum field theory. We want to mention some examples.

If there is a representation of the translation group of positive energy which leaves the vacuum Ω invariant and if W is a wedge and l a lightlike vector such that $W + l \subset W$, then the algebras associated with the wedges W and $W + l$ must form a half-sided modular inclusion. This shows that, in general, the split property cannot hold for pairs of lightlike separated wedge algebras, even if the vacuum state is reducible or if we consider two-dimensional theories (see also [14]).

If Ω is cyclic and separating for the algebras associated with light cones, then the modular groups of these algebras exist and must act on the translation group as dilatations (see Theorem 2.1.1). Hence, we obtain here half-sided modular inclusions again, i.e., a pair of two of these algebras can never be split.

Finally we want to remark that we have found state-independent conditions a net of von Neumann algebras has to satisfy if it admits a translation invariant state of positive energy.

We consider a half-sided modular inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$ and an intermediate von Neumann algebra $\mathcal{L}$, i.e., $\mathcal{N} \subset \mathcal{L} \subset \mathcal{M}$. We denote by $\Delta_{\mathcal{N}}, \Delta_{\mathcal{L}}, \Delta_{\mathcal{M}}$ and $J_{\mathcal{N}}, J_{\mathcal{L}}, J_{\mathcal{M}}$ the modular operators and modular conjugations of $\mathcal{N}, \mathcal{L}$, and $\mathcal{M}$ with respect to Ω. These operators are well-defined, since Ω is cyclic and separating for $\mathcal{N}$ and $\mathcal{M}$, i.e., Ω is also cyclic for $\mathcal{L} \supset \mathcal{N}$ and separating for $\mathcal{L} \subset \mathcal{M}$.

Proposition 2.3.1: *Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a $\pm$half-sided modular inclusion and $\mathcal{L}$ a von Neumann algebra with $\mathcal{N} \subset \mathcal{L} \subset \mathcal{M}$. $\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}$ converges for $t \to \mp\infty$ strongly to an isometry $U = J_{\mathcal{L}} U J_{\mathcal{M}}$.*

Proof: We assume that $(\mathcal{N} \subset \mathcal{M}, \Omega)$ is a -half-sided modular inclusion. The other case follows similarly. There is a strongly continuous group $V(t)$ with positive generator such that

$$V(1 - e^{-2\pi t}) \;=\; \Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it}$$

[40, 42] and

$$\Delta_{\mathcal{N}}^{it} J_{\mathcal{N}} J_{\mathcal{M}} \Delta_{\mathcal{M}}^{-it} \;=\; V(1 - e^{-2\pi(t+i/2)}) \;=\; V(1 + e^{-2\pi t}) \tag{2.3.1}$$

[16].

Let φ be an analytic vector of $V(z)$. Then $V(z)\varphi$ is bounded for bounded z. Note that, since $\mathcal{L} \subset \mathcal{M}$, $\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}$ has a bounded, strongly continuous extension $A(z)$ to the strip $S_+ = \{z \in \mathbf{C} \,|\, 0 \leq \mathrm{Im} z \leq \frac{1}{2}\}$, which is analytic in the interior and such that $A(t+i/2) = \Delta_{\mathcal{L}}^{it} J_{\mathcal{L}} J_{\mathcal{M}} \Delta_{\mathcal{M}}^{-it}$ for $t \in \mathbf{R}$ [16]. Similarly, it can be proven that $\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{N}}^{-it}$ has a bounded, strongly continuous extension $B(z)$ to the strip $S_- = \{z \in \mathbf{C} \,|\, 0 \geq \mathrm{Im} z \geq -\frac{1}{2}\}$, which is analytic in the interior and such that $B(t - i/2) = \Delta_{\mathcal{L}}^{it} J_{\mathcal{L}} J_{\mathcal{N}} \Delta_{\mathcal{N}}^{-it}$ for $t \in \mathbf{R}$ (note $\mathcal{M} \supset \mathcal{L}$ but $\mathcal{N} \subset \mathcal{L}$). The function

$$A(t)\varphi \;=\; \Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{N}}^{-it}\Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it}\varphi \;=\; B(t)V(1 - e^{-2\pi t})\varphi, \quad t \in \mathbf{R}, \tag{2.3.2}$$

possesses a strongly continuous extension to $S_+ \cup S_-$ given by $A(z)\varphi$ on S_+ and by $B(z)V(1-e^{-2\pi z})\varphi$ on S_-. We will call this extension $f(z)$. According to the reflection principle and the fact that weak and strong analyticity are equivalent (see the proof of Theorem 2.5.21 in [11]), $f(z)$ is analytic in the interior of $S_+ \cup S_-$. Note that $f(z)$ is bounded for bounded $V(1-e^{-2\pi z})\varphi$, resp. for bounded $1-e^{-2\pi z}$. We will show

$$f(t+i/2) \;=\; f(t-i/2) \quad \text{for} \quad t \in \mathbf{R}. \tag{2.3.3}$$

This follows from

$$\begin{aligned} f(t+i/2) &= A(t+i/2)\varphi \\ &= \Delta_{\mathcal{L}}^{it} J_{\mathcal{L}} J_{\mathcal{M}} \Delta_{\mathcal{M}}^{-it}\varphi \end{aligned}$$

and

$$\begin{aligned} f(t-i/2) &= B(t-i/2)V(1-e^{-2\pi(t-i/2)})\varphi \\ &= B(t-i/2)V(1+e^{-2\pi t})\varphi \\ &\overset{(2.3.1)}{=} (\Delta_{\mathcal{L}}^{it} J_{\mathcal{L}} J_{\mathcal{N}} \Delta_{\mathcal{N}}^{-it})(\Delta_{\mathcal{N}}^{it} J_{\mathcal{N}} J_{\mathcal{M}} \Delta_{\mathcal{M}}^{-it})\varphi \\ &= \Delta_{\mathcal{L}}^{it} J_{\mathcal{L}} J_{\mathcal{M}} \Delta_{\mathcal{M}}^{-it}\varphi. \end{aligned}$$

(2.3.3) implies that $g(z) = f(-\frac{1}{2\pi}\ln z)$ is well-defined for $z \neq 0$ if we choose $\ln z$ such that $-\pi \leq \operatorname{Im}\ln z \leq \pi$. We want to show that $g(z)$ is analytic for $z \neq 0$ and then that the singularity at $z = 0$ is removable. $g(z)$ is weakly continuous for $z \neq 0$ and analytic in $\{z \in \mathbf{C}\backslash\{0\} \,|\, \operatorname{Im}\ln z \neq \pm\pi\} = \mathbf{C}\backslash\mathbf{R}_-$. According to the reflection principle and the fact that weak and strong analyticity are equivalent, $g(z)$ is even analytic for $z \neq 0$. We have shown above that $f(z)$ is bounded for bounded $1-e^{-2\pi z}$, i.e. $g(z) = f(-\frac{1}{2\pi}\ln z)$ is bounded for bounded $1-e^{-2\pi(-\frac{1}{2\pi}\ln z)} = 1-z$ and, in particular, for $0 < |z| < 1$. For any vector ψ, $(\psi, g(z))$ is analytic in $z \neq 0$. This function can even be extended to an entire function, since singularities of bounded analytic functions are removable. However, weak and strong analyticity are equivalent implying that the singularity of $g(z)$ at $z = 0$ is removable.

We have shown that $g(z) = f(-\frac{1}{2\pi}\ln z)$ converges strongly for $z \to 0$, i.e., $f(z) = g(e^{-2\pi z})$ converges for $e^{-2\pi z} \to 0$, resp. for $\mathrm{Re} z \to \infty$. Therefore, $f(t) = \Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}\varphi$, $t \in \mathbf{R}$, converges to $\lim_{z\to 0} g(z)$ for $t \to \infty$ and $\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}$ converges strongly to an isometry U for $t \to \infty$. $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f(t + i/2)$ shows $U = J_{\mathcal{L}} U J_{\mathcal{M}}$.

If U in Proposition 2.3.1 is unitary, then it is not difficult to prove $\mathcal{L} = U\mathcal{M}U^*$, i.e. that $\mathcal{L}$ is of type III_1 if $\mathcal{N} \neq \mathcal{M}$ [10]. In the following we will show that, even if U is not unitary, $\mathcal{L}$ cannot be a type I factor for $\mathcal{N} \neq \mathcal{M}$. In 2.5 we will give an example which shows that the isometry U in Proposition 2.3.1 is not necessarily unitary.

We denote by P the projection UU^* (note $U^*U = 1$). There are examples which show that $P \neq 1$ is possible (see Section 2.5).

Lemma 2.3.2: *Under the assumptions of Proposition 2.3.1 we denote by* $\mathcal{L}_0$ *the von Neumann algebra* $\{L \in \mathcal{L} \mid L = PLP + (1-P)L(1-P)\}$. $\mathcal{M}$ *and* $\mathcal{L}_0$ *are *isomorphic.*

Proof: We want to show that there is a *isomorphism $\varphi : \mathcal{M} \mapsto \mathcal{L}_0$ given by $\varphi(M)\Omega = UM\Omega$. Note that, for $M \in \mathcal{M}$,

$$\begin{aligned}
J_{\mathcal{L}}\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}J_{\mathcal{M}}MJ_{\mathcal{M}}\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{L}}^{-it}J_{\mathcal{L}} &\in J_{\mathcal{L}}\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}\mathcal{M}'\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{L}}^{-it}J_{\mathcal{L}} \\
&\subset J_{\mathcal{L}}\Delta_{\mathcal{L}}^{it}\mathcal{M}'\Delta_{\mathcal{L}}^{-it}J_{\mathcal{L}} \\
&\subset J_{\mathcal{L}}\Delta_{\mathcal{L}}^{it}\mathcal{L}'\Delta_{\mathcal{L}}^{-it}J_{\mathcal{L}} \\
&= \mathcal{L}.
\end{aligned}$$

Let $L \in \mathcal{L}$ be a weak limit point of $\{J_{\mathcal{L}}\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}J_{\mathcal{M}}MJ_{\mathcal{M}}\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{L}}^{-it}J_{\mathcal{L}}\}$ for $t \to \mp\infty$. L is unique, since Ω is separating for $\mathcal{L}$ and

$$L\Omega = J_{\mathcal{L}}UJ_{\mathcal{M}}M\Omega = UM\Omega.$$

Hence, $\varphi(M)\Omega = UM\Omega$ defines a *map into $\mathcal{L}$.

Note that $\{J_{\mathcal{M}}\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{L}}^{-it}J_{\mathcal{L}}P\}$ converges weakly, hence strongly, to $U^*P = U^*$ for $t \to \mp\infty$, i.e., $\{J_{\mathcal{L}}\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}J_{\mathcal{M}}MJ_{\mathcal{M}}\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{L}}^{-it}J_{\mathcal{L}}P\}$ converges strongly to UMU^*P for $t \to \mp\infty$. We have shown $\varphi(M)P = UMU^*P = UMU^*$. It follows $P\varphi(M) = P\varphi(M^*)^* = (\varphi(M^*)P)^* = UMU^* = \varphi(M)P$, i.e., φ maps $\mathcal{M}$ into $\mathcal{L}_0$.

$$\begin{aligned}\varphi(M_1M_2)\Omega &= UM_1M_2\Omega\\ &= UM_1U^*UM_2\Omega\\ &= UM_1U^*PUM_2\Omega\\ &= \varphi(M_1)\varphi(M_2)\Omega,\end{aligned}$$

for $M_1, M_2 \in \mathcal{M}$, shows that φ is a *homomorphism, since Ω is separating for $\mathcal{L}_0$.

Define $\psi : \mathcal{L}_0 \mapsto \mathcal{M}$ by $\psi(L) = U^*LU = U^*PLPU$. $\{\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{L}}^{-it}L\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it} \in \mathcal{M}\}$ converges weakly to U^*LU for $t \to \mp\infty$, since $\{\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}\}$ converges strongly to U and $\{\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{L}}^{-it}\}$ converges weakly to U^*. Hence, ψ is indeed a *homomorphism into $\mathcal{M}$. $\psi = \varphi^{-1}$ follows from

$$P(\varphi \circ \psi(L))P = P\varphi(U^*LU)P = U(U^*LU)U^* = PLP,$$

resp. $\varphi \circ \psi(L) = L$, for $L \in \mathcal{L}_0$ (note that $PAP = 0$ for $A \in \mathcal{L}_0$ implies $A = 0$, since $AP = PAP$ and Ω is separating for $\mathcal{L}_0$) and

$$\psi \circ \varphi(M) = U^*\varphi(M)U = U^*P\varphi(M)PU = U^*PUMU^*PU = M,$$

for $M \in \mathcal{M}$.

Proposition 2.3.3: *The assumptions in Proposition 2.3.1 imply*

$$U^*\mathcal{L}U = U^*\mathcal{L}_0U = \mathcal{M} \quad \text{and} \quad U\mathcal{M}U^* = P\mathcal{L}P = P\mathcal{L}_0P.$$

Proof: The proof of the preceding lemma implies

$$U^*\mathcal{L}U \supset U^*\mathcal{L}_0U = \mathcal{M}.$$

Note that we obtain $U\mathcal{M}U^* = P\mathcal{L}P = P\mathcal{L}_0P$ if we adjoin $U^*\mathcal{L}U = U^*\mathcal{L}_0U = \mathcal{M}$ by U. Hence, it is sufficient to show

$$U^*\mathcal{L}U \subset \mathcal{M}.$$

$\{\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it}\}$ converges strongly to U and $\{\Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{L}}^{-it}\}$ converges weakly to U^* for $t \to \mp\infty$. Thus,

$$\begin{aligned} \Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{L}}^{-it}L\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it} &\in \Delta_{\mathcal{M}}^{it}\mathcal{L}\Delta_{\mathcal{M}}^{-it} \\ &\subset \Delta_{\mathcal{M}}^{it}\mathcal{M}\Delta_{\mathcal{M}}^{-it} \\ &= \mathcal{M} \end{aligned}$$

converges weakly to $U^*LU \in \mathcal{M}$, for $L \in \mathcal{L}$ and $t \to \mp\infty$.

Theorem 2.3.4: *Let $(\mathcal{N} \subset \mathcal{M}, \Omega)$ be a nontrivial half-sided modular inclusion. The pair $(\mathcal{N}, \mathcal{M})$ is not split.*

Proof: We will use the same notations as above and we assume that $\mathcal{L}$ is a type I factor. Let $Q \in \mathcal{L}\backslash\{0\}$ be a minimal projection, i.e., $QRQ = \lambda Q$ for all $R \in \mathcal{L}$ and suitable $\lambda \in \mathbf{R}$. $\omega(PQP) = \| Q\Omega \|^2 \neq 0$ entails $PQP \neq 0$. There is a spectral projection $0 \neq S \in P\mathcal{L}_0P$ of $PQP \in P\mathcal{L}P = P\mathcal{L}_0P$ such that $PQP \geq cS$ for a $c > 0$

($P\mathcal{L}_0P$ is a von Neumann algebra). Since $\mathcal{M}$, resp. $P\mathcal{L}_0P$, is of type III [10], there is a decreasing net of nontrivial projections $\{S_\alpha \in P\mathcal{L}_0P\}$ which weakly converges to 0 and such that $S_\alpha \leq S$ (use Zorn's lemma and note that $P\mathcal{L}_0P$ is infinite). The homomorphism $\phi : \mathcal{L}_0 \mapsto P\mathcal{L}_0P$ given by $\phi(L) = LP$, for $L \in \mathcal{L}_0$, is bijective, since Ω is separating for $\mathcal{L}_0$. Hence, we can define projections $T_\alpha \in \mathcal{L}_0$ by $T_\alpha P = S_\alpha$. $\{T_\alpha\}$ converges weakly to 0, as well. The minimality of Q and

$$\begin{aligned} \| QT_\alpha Q \| &= \| T_\alpha Q \|^2 \geq \| PT_\alpha QP \|^2 = \| S_\alpha PQP \|^2 \\ &\geq \| S_\alpha PQPS_\alpha \|^2 \geq \| S_\alpha(cS)S_\alpha \|^2 = \| cS_\alpha \|^2 = c^2, \end{aligned}$$

imply $QT_\alpha Q \geq c^2 Q$, in contradiction to the weak convergence of $\{T_\alpha\}$ to 0.

2.4 Additivity

We will apply the following theorem to algebraic quantum field theory.

Theorem 2.4.1: *Let Ω be cyclic and separating with respect to the von Neumann algebras $\mathcal{M}$, $\mathcal{N}_+$, and $\mathcal{N}_-$, which act on the Hilbert space $\mathcal{H}$. Assume that $(\mathcal{N}_+ \subset \mathcal{M}, \Omega)$ is a +half-sided modular inclusion and that $(\mathcal{N}_- \subset \mathcal{M}, \Omega)$ is a -half-sided modular inclusion. If Ω is cyclic for $\mathcal{N}_+ \cap \mathcal{N}_-$, then $\mathcal{M} = \mathcal{N}_+ \vee \mathcal{N}_-$.*

Proof: see [23]

In 2.5, we will show that the conclusion in Theorem 2.4.1 can fail if Ω is not cyclic for $\mathcal{N}_+ \cap \mathcal{N}_-$. We can prove the dual of Theorem 2.4.1.

Corollary 2.4.2: *Let Ω be cyclic and separating with respect to the von Neumann algebras $\mathcal{M}$, $\mathcal{N}_+$, and $\mathcal{N}_-$, which act on a Hilbert space $\mathcal{H}$. Assume that $(\mathcal{M} \subset \mathcal{N}_+, \Omega)$*

is a +half-sided modular inclusion and that $(\mathcal{M} \subset \mathcal{N}_-, \Omega)$ *is a -half-sided modular inclusion. If* Ω *is separating for* $\mathcal{N}_+ \vee \mathcal{N}_-$*, then* $\mathcal{M} = \mathcal{N}_+ \cap \mathcal{N}_-$.

Proof: Consider the algebras $\mathcal{M}'$, $\mathcal{N}_+'$, and $\mathcal{N}_-'$. It is not difficult to show that $(\mathcal{N}_+' \subset \mathcal{M}', \Omega)$ and $(\mathcal{N}_-' \subset \mathcal{M}', \Omega)$ are half-sided modular inclusions. Furthermore, Ω is cyclic for $(\mathcal{N}_+ \cup \mathcal{N}_-)' = \mathcal{N}_+' \cap \mathcal{N}_-'$, since it is separating for $\mathcal{N}_+ \vee \mathcal{N}_-$. Theorem 2.4.1 implies $\mathcal{M}' = (\mathcal{N}_+' \cup \mathcal{N}_-')'' = (\mathcal{N}_+ \cap \mathcal{N}_-)'$, resp. $\mathcal{M} = \mathcal{N}_+ \cap \mathcal{N}_-$.

The following theorem follows easily from Theorem 2.4.1 and Corollary 2.4.2. We consider coherent wedges, i.e., wedges which we can obtain from each other by translations.

Theorem 2.4.3: *Let* $\mathcal{W}$ *be the set of wedges in Minkowski space. Assume that* $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ *is a net of von Neumann algebras, which act on a Hilbert space* $\mathcal{H}$*. Let* Ω *be a cyclic and separating vector of the algebras* $\mathcal{R}(W)$*. If there is a strongly continuous representation of the group of translations of positive energy on* $\mathcal{H}$ *which acts geometrically correctly on the set of wedge algebras and leaves* Ω *invariant, then*

$$\mathcal{R}(W_1 \vee W_2) = \mathcal{R}(W_1) \vee \mathcal{R}(W_2), \quad \mathcal{R}(W_1 \cap W_2) = \mathcal{R}(W_1) \cap \mathcal{R}(W_2),$$

for coherent wedges $W_1, W_2 \in \mathcal{W}$*. We denote by* $W_1 \vee W_2$ *the causal closure of* $W_1 \cup W_2$.

Proof: $(\mathcal{R}(W_1) \subset \mathcal{R}(W_1 \vee W_2), \Omega)$ is a $\pm$half-sided modular inclusion and $(\mathcal{R}(W_2) \subset \mathcal{R}(W_1 \vee W_2), \Omega)$ is a $\mp$half-sided modular inclusion. We can apply Theorem 2.4.1, since Ω is cyclic for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2) \supset \mathcal{R}(W_1 \cap W_2)$ (note that $W_1 \cap W_2$

is a wedge).

We denote by $K_{a,b}$ the double cone with vertices $a, b \in \mathbf{R}^4$, where $a - b$ is a timelike vector.

Theorem 2.4.4: *Under the assumptions of Theorem 2.4.3, we set*

$$\mathcal{R}(K) = \cap_{K \subset W \in \mathcal{W}} \mathcal{R}(W)$$

for double cones K. If $K_{a,b}, K_{c,d}$ are double cones such that $c \in K_{a,b}$, $b \in K_{c,d}$, and Ω is cyclic for $\mathcal{R}(K_{c,b}) = \mathcal{R}(K_{a,b} \cap K_{c,d})$, then

$$\mathcal{R}(K_{a,b} \cap K_{c,d}) = \mathcal{R}(K_{a,b}) \cap \mathcal{R}(K_{c,d}).$$

In two dimensions this relation even holds for arbitrary pairs of double cones with nontrivial intersection.

Proof: Note $K_{a,b} \cap K_{c,d} = K_{c,b}$ and use

$$\mathcal{R}(W_1 \cap W_2) = \mathcal{R}(W_1) \cap \mathcal{R}(W_2) \tag{2.4.1}$$

for coherent wedges $W_1, W_2 \in \mathcal{W}$. In two dimensions each double cone algebra is the intersection of two wedge algebras. The claim can be proven in this case if we use (2.4.1) and isotony for wedge algebras (there are several cases to consider depending on the relative position of the double cones).

2.5 Conformal Quantum Field Theory

First we show how the results of Section 2.4 can be applied to derive additivity for interval algebras in algebraic quantum field theories on the circle. We consider a net of von Neumann algebras $\mathcal{R}(I)$ indexed by proper open intervals $I \subset S^1$ and acting on a Hilbert space $\mathcal{H}$. We assume isotony and that the vector $\Omega \in \mathcal{H}$ induces a vacuum state, which is cyclic and separating for the algebras $\mathcal{R}(I)$. The latter assumption can be derived from locality and the positivity of energy. We denote by Δ_I the modular operators of the pairs $(\mathcal{R}(I), \Omega)$. Note that we obtain half-sided modular inclusions $(\mathcal{R}(I_1) \subset \mathcal{R}(I_2), \Omega)$ if $\emptyset \neq I_1 \subset I_2$ are proper open intervals with a common endpoint [7]. The following theorem was already proven in [23].

Theorem 2.5.1: *Let $I_1, I_2 \subset S^1$ be open intervals with nontrivial intersection such that $I_1 \cup I_2$ is a proper interval. Then*

$$\mathcal{R}(I_1 \cup I_2) = \mathcal{R}(I_1) \vee \mathcal{R}(I_2), \quad \mathcal{R}(I_1 \cap I_2) = \mathcal{R}(I_1) \cap \mathcal{R}(I_2)$$

Proof: Theorem 2.4.1 and Corollary 2.4.2

Remark: The conclusion in Theorem 2.4.1, resp. Corollary 2.4.2, is in general not true if Ω is not cyclic for $\mathcal{N}_+ \cap \mathcal{N}_-$, resp. if Ω is not separating for $\mathcal{N}_+ \vee \mathcal{N}_-$.

Proof: Consider open, non-empty intervals $I_1, I_2 \subset S^1$ such that $\overline{I_1} \cap \overline{I_2} = \emptyset$. There are proper intervals $J_1 \neq J_2$ which contain I_1 and I_2 and have common endpoints with I_1 and I_2, i.e., such that $(\mathcal{R}(I_k) \subset \mathcal{R}(J_l), \Omega)$ are half-sided modular inclusions for $k, l = 1, 2$. However,

$$\mathcal{R}(J_1) \;=\; \mathcal{R}(I_1) \vee \mathcal{R}(I_2) \;=\; \mathcal{R}(J_2)$$

is impossible for a nontrivial net.

Remark: The isometry U in Proposition 2.3.1 is not necessarily unitary.

Proof: We assume that the isometries in Proposition 2.3.1 are unitary and derive a contradiction in this setting of conformal quantum field theory. Consider two proper open intervals I_1, I_2 such that $I_1 \cap I_2$ consists of two disjoint, non-empty intervals A, B. We denote by $\Delta_{A\cup B}$ the modular operator of $(\mathcal{R}(A) \vee \mathcal{R}(B), \Omega)$. Using the $\pm$half-sided modular inclusion $(\mathcal{R}(A) \subset \mathcal{R}(I_1), \Omega)$ and the $\mp$half-sided modular inclusion $(\mathcal{R}(B) \subset \mathcal{R}(I_1), \Omega)$ we can conclude from Proposition 2.3.1 that $\Delta_{A\cup B}^{it}\Delta_{I_1}^{-it}$ converges strongly for $t \to \infty$ and for $t \to -\infty$. Similarly, it follows that $\Delta_{A\cup B}^{it}\Delta_{I_2}^{-it}$ converges strongly for $t \to \infty$ and $t \to -\infty$. Since we assume that the limit points are unitary, this implies the strong convergence of

$$(\Delta_{A\cup B}^{it}\Delta_{I_1}^{-it})^*\Delta_{A\cup B}^{it}\Delta_{I_2}^{-it} = \Delta_{I_1}^{it}\Delta_{I_2}^{-it}$$

for $t \to \infty$ and for $t \to -\infty$. Choose a proper interval $I_3 \supset I_1$ such that $(\mathcal{R}(I_1) \subset \mathcal{R}(I_3), \Omega)$ forms a nontrivial half-sided modular inclusion. By using the same arguments as above, we can show the strong convergence of $\Delta_{I_2}^{it}\Delta_{I_3}^{-it}(I_2 \cap I_3 \supset I_2 \cap I_1$ consists of two disjoint, non-empty intervals), resp. of

$$(\Delta_{I_1}^{it}\Delta_{I_2}^{-it})(\Delta_{I_2}^{it}\Delta_{I_3}^{-it}) = \Delta_{I_1}^{it}\Delta_{I_3}^{-it},$$

for $t \to \infty$ and $t \to -\infty$. Since $(\mathcal{R}(I_1) \subset \mathcal{R}(I_3), \Omega)$ forms a half-sided modular inclusion, there is a group $V(t)$ with positive generator such that either

$$\Delta_{I_1}^{it}\Delta_{I_3}^{-it} = V(e^{2\pi t} - 1)$$

or

$$\Delta_{I_1}^{it}\Delta_{I_3}^{-it} = V(1 - e^{-2\pi t})$$

[40, 42]. In any case we can conclude that $V(s)$ converges strongly for $s \to \infty$ or for $s \to -\infty$. Since we assume that the limit is a unitary operator, we obtain the strong

convergence for $s \to \infty$ and for $s \to -\infty$, i.e.,

$$V(a) \;=\; \lim_{s\to\infty} V(s+a)V(-s) \;=\; \lim_{s\to\infty} V(s+a)(\lim_{s\to\infty} V(s))^* \;=\; 1$$

for all $a \in \mathbf{R}$, resp. that

$$\mathcal{R}(I_1) \;=\; V(\pm 1)\mathcal{R}(I_3)V(\mp 1) \;=\; \mathcal{R}(I_3),$$

which is impossible for a nontrivial net.

Now we will consider conformal quantum field theories in at least two dimensions.

Lemma 2.5.2: *Suppose that $\mathcal{K}$ is a set of regions in Minkowski space which contains the set of double cones and which is invariant with respect to the identity component of the conformal group. Let $\{\mathcal{R}(K)\}_{K\in\mathcal{K}}$ be a net of von Neumann algebras acting on a Hilbert space $\mathcal{H}$. We assume that Ω is cyclic and separating for wedge algebras and that there is a representation of positive energy of the identity component of the conformal group which acts geometrically correctly on the net and leaves the vector Ω invariant. Let $K_{a,b}$ be a double cone and $c \in \overline{K_{a,b}} \backslash \{a, b\}$. Then $(\mathcal{R}(K_{a,c}) \subset \mathcal{R}(K_{a,b}), \Omega)$ is a -half-sided modular inclusion and $(\mathcal{R}(K_{c,b}) \subset \mathcal{R}(K_{a,b}), \Omega)$ is a +half-sided modular inclusion.*

Proof: We only show that $(\mathcal{R}(K_{a,c}) \subset \mathcal{R}(K_{a,b}), \Omega)$ is a -half-sided modular inclusion, since the other case follows similarly. We can assume $a = 0$. The conformal group leaves the metric $\mathrm{diag}(1, -1, -1, -1, -1, 1)$ invariant (see e.g. [26, I.2]). It is not difficult to show that $\mathrm{diag}(1, -1, -1, -1, -1, 1)$ is an element of the identity component of the conformal group, which acts on $\mathbf{R}^4$ as $S : x \mapsto Tx/(x, x)$, where the metric on

Minkowski space, resp. $(\cdot,\cdot)$, is given by $\mathrm{diag}(1,-1,-1,-1)$ and where we denote the time inversion by T. Let V_z be the forward light cone with vertex z. S maps $K_{0,b}$ and $K_{0,c}$ onto the forward light cones $V_{S(b)}$ and $V_{S(c)}$ (consider the boundary of the double cones and note that S maps lightlike x, resp. $\partial(-V_0)$, to infinity and $\partial K_{0,b}\backslash\partial(-V_0)$, resp. $\partial K_{0,c}\backslash\partial(-V_0)$, to $\partial V_{S(b)}$, resp. $\partial V_{S(c)}$). Since $V_{S(c)} \subset V_{S(b)}$, there is a translation by a non-spacelike vector y which maps $V_{S(b)}$ onto $V_{S(c)}$. According to [7], the modular group of $\mathcal{R}(V_{S(b)})$ acts on the one-parameter group of translations in the direction of y as dilatations and $(\mathcal{R}(V_{S(c)}) \subset \mathcal{R}(V_{S(b)}),\Omega)$, resp. $(\mathcal{R}(K_{a,c}) \subset \mathcal{R}(K_{a,b}),\Omega)$, is a -half-sided modular inclusion.

Theorem 2.5.3: *Suppose that $\mathcal{K}$ is a set of regions in Minkowski space which contains the set of double cones and which is invariant with respect to the identity component of the conformal group. Let $\{\mathcal{R}(K)\}_{K\in\mathcal{K}}$ be a net of von Neumann algebras acting on a Hilbert space $\mathcal{H}$. We assume that Ω is cyclic and separating for wedge algebras and that there is a representation of the identity component of the conformal group of positive energy which acts geometrically correctly on the net and leaves the vector Ω invariant. If $K_{a,b}, K_{c,d}$ are double cones such that $c \in K_{a,b}$ and $b \in K_{c,d}$, then*

$$\mathcal{R}(K_{a,b}) \vee \mathcal{R}(K_{c,d}) = \mathcal{R}(K_{a,d}) \quad \text{and} \quad \mathcal{R}(K_{a,b}) \cap \mathcal{R}(K_{c,d}) = \mathcal{R}(K_{c,b})$$

Proof: Ω is cyclic and separating for wedge algebras and hence for double cone algebras. $(\mathcal{R}(K_{a,b}) \subset \mathcal{R}(K_{a,d}),\Omega)$ is a $\pm$half-sided modular inclusion and $(\mathcal{R}(K_{c,d}) \subset \mathcal{R}(K_{a,d}),\Omega)$ is a $\mp$half-sided modular inclusion. Theorem 2.4.1 entails $\mathcal{R}(K_{a,b}) \vee \mathcal{R}(K_{c,d}) = \mathcal{R}(K_{a,d})$. The other claim follows similarly.

Corollary 2.5.4: *Suppose that $\mathcal{K}$ is a set of regions in Minkowski space which contains the set of double cones and which is invariant with respect to the identity component of the conformal group. Let $\{\mathcal{R}(K)\}_{K\in\mathcal{K}}$ be a net of von Neumann algebras acting on a Hilbert space $\mathcal{H}$. We assume that Ω is cyclic and separating for wedge algebras and that there is a representation of the identity component of the conformal group of positive energy which acts geometrically correctly on the net and leaves the vector Ω invariant. If $\mathcal{O} \in \mathcal{K}$ contains a timelike curve joining $a, b \in \mathcal{O}$, then*

$$\mathcal{R}(K_{a,b}) \subset \mathcal{R}(\mathcal{O}).$$

Proof: Let $\gamma : [0,1] \mapsto \mathcal{O}$ be a timelike curve with $\gamma(0) = a$ and $\gamma(1) = b$. Cover $\gamma((0,1))$ by a finite number of double cones which are contained in $\mathcal{O}$ and such that their vertices lie on $\gamma([0,1])$ (in particular, a, resp. b, is a vertex of one double cone). Then we can repeatedly apply Theorem 2.5.3.

Theorems which are similar to the preceding corollary were already derived for Wightman fields [3, 6]. The proofs could also be applied to our setting. However, we would have to assume in addition an additivity assumption on the net.

CHAPTER 3
GEOMETRIC MODULAR ACTION ON MINKOWSKI SPACE

In this chapter we want to consider nets of von Neumann algebras over Minkowski space. We will assume suitable conditions which allow the derivation of spacetime symmetry groups and their corresponding representations with the help of the modular objects of the von Neumann algebras. Motivated by the results of Bisognano and Wichmann [4, 5] we choose the set of wedges $\mathcal{W}$ and consider a net of von Neumann algebras $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, which satisfies the CGMA. We obtain $\mathcal{W}$ by applying elements of the Poincaré group to the standard wedge

$$W_R \equiv \{x \in \mathbf{R}^4 \,|\, x_1 > |x_0|\}.$$

In addition, we will assume further purely algebraic conditions, which will imply that the modular conjugations of the pairs $(\mathcal{R}(W), \Omega)$ act as point transformations.

Strong Condition of Geometric Modular Action: Let $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ be a net of von Neumann algebras, which act on a Hilbert space $\mathcal{H}$ and let $\Omega \in \mathcal{H}$. $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}}, \Omega)$ satisfies the strong CGMA if the following conditions are satisfied.

(i) $W \mapsto \mathcal{R}(W)$ *is an order-preserving bijection.*

(ii) Ω *is cyclic and separating for* $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$, $W_1, W_2 \in \mathcal{W}$, *if* $W_1 \cap W_2 \neq 0$.

(iii) If Ω *is cyclic and separating for* $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$, $W_1, W_2 \in \mathcal{W}$, *then* $\overline{W_1} \cap \overline{W_2} \neq \emptyset$.

(iv) *The adjoint action of the modular conjugations* J_{W_1}, $W_1 \in \mathcal{W}$, *of the pairs* $(\mathcal{R}(W_1), \Omega)$ *leaves the net* $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ *invariant, i.e.*

$$J_{W_1} \mathcal{R}(W) J_{W_1} = \mathcal{R}(W_2) \quad \forall W, W_1 \in \mathcal{W}$$

and suitable $W_2 \in \mathcal{W}$.

It is remarkable that these natural conditions can be stated on arbitrary spacetimes. Hence, we consider the strong CGMA as a selection criterion for physically interesting states on general spacetimes.

Note that the modular conjugations J_W of $(\mathcal{R}(W), \Omega)$ are well-defined (see (ii)). The lack of symmetry in conditions (ii) and (iii) is introduced in order to admit certain massless theories, in which Ω can be cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$ even if $W_1 \cap W_2 = \emptyset$ (in these examples, this can occur for unbounded $\overline{W_1} \cap \overline{W_2}$).

The strong CGMA entails that the modular conjugations of wedge algebras act as Poincaré transformations (Theorem 3.1.3). If their action is transitive on the net, we can recover the same action of the modular conjugations as in [4, 5]. With the help of an additional assumption we can derive then a strongly continuous representation of the proper Poincaré group, which acts geometrically correctly on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ (Theorem 3.5.3.2, Theorem 3.5.3.4).

We will also consider other versions of the strong CGMA, i.e., instead of (iv), we will assume that only even products of modular conjugations, resp. that the elements of the modular groups of $(\mathcal{R}(W), \Omega)$, have a geometric action.

We will repeatedly use that an (anti)unitary U which satisfies

$$U \mathcal{R}(W) U^* = \mathcal{R}(W) \quad \text{and} \quad U\Omega = \Omega$$

commutes with the modular objects of the pair $(\mathcal{R}(W), \Omega)$. This uniqueness of the modular objects follows easily from their definition. In particular, we can conclude that elements of $\mathcal{J}$ whose adjoint action leaves the net pointwise fixed lie in the center of $\mathcal{J}$. Hence, $\mathcal{J}$ is a central extension of a subgroup of the Poincaré group, since the elements of $\mathcal{J}$ act as Poincaré transformations (Theorem 3.1.3).

In the following we will denote by $\mathcal{P}$ the Poincaré group, by $\mathcal{P}_+$ the proper Poincaré group, and by $\mathcal{P}_+^\uparrow$ its identity component. Let $\mathcal{L}_+$ and $\mathcal{L}_+^\uparrow$ be the corresponding subgroups of the Lorentz group $\mathcal{L}$.

3.1 Bijections of Wedges

We will consider bijective maps $\tau : \mathcal{W} \mapsto \mathcal{W}$ which satisfy the following properties:

(A) $W_1, W_2 \in \mathcal{W}$ satisfy $W_1 \cap W_2 = \emptyset$ if and only if $\tau(W_1) \cap \tau(W_2) = \emptyset$.

(B) $W_1, W_2 \in \mathcal{W}$ satisfy $W_1 \subset W_2$ if and only if $\tau(W_1) \subset \tau(W_2)$.

We will use the notations

$$l_{1\pm} = (1, \pm 1, 0, 0)^{\mathrm{T}}, \quad l_{2\pm} = (1, 0, \pm 1, 0)^{\mathrm{T}}, \quad l_{3\pm} = (1, 0, 0, \pm 1)^{\mathrm{T}}.$$

For linearly independent lightlike vectors l_1, l_2 in the forward lightcone and arbitrary $a \in \mathbf{R}^4$, wedges can be defined by

$$W[l_1, l_2, a] \;=\; \{\alpha l_1 + \beta l_2 + l^\perp + a \,|\, \alpha > 0, \beta < 0, (l^\perp, l_1) = (l^\perp, l_2) = 0\},$$

where the scalar product means the Minkowski space product.

The following two lemmas will justify our interest in bijections which satisfy (A) and (B).

Lemma 3.1.1: *Let τ be a bijection on $\mathcal{W}$ which fulfills*

$$\overline{W_1} \cap \overline{W_2} = \emptyset \quad \Rightarrow \quad \tau(W_1) \cap \tau(W_2) = \emptyset \quad \forall W_1, W_2 \in \mathcal{W} \tag{3.1.1}$$

and

$$W_1 \subset W_2 \quad \Leftrightarrow \quad \tau(W_1) \subset \tau(W_2) \quad \forall W_1, W_2 \in \mathcal{W}. \tag{3.1.2}$$

Then

$$W_1 \cap W_2 = \emptyset \quad \Rightarrow \quad \tau(W_1) \cap \tau(W_2) = \emptyset \quad \forall W_1, W_2 \in \mathcal{W}$$

Proof: Let $W_1, W_2 \in \mathcal{W}$ be such that $W_1 \cap W_2 = \emptyset$ and $\overline{W_1} \cap \overline{W_2} \neq 0$. We have to show $\tau(W_1) \cap \tau(W_2) = \emptyset$. Let $N \subset \overline{W_R} \backslash W_R$ be convex. It is easy to see that either

$$N \cap \overline{W[l_{1+}, l_{1-}, \lambda l_{1+}]} = \emptyset \quad \forall \lambda > 0 \quad \text{or} \quad N \cap \overline{W[l_{1+}, l_{1-}, \lambda l_{1-}]} = \emptyset \quad \forall \lambda < 0$$

(the vectors in $\mathbf{R}^2$ formed by the first two components of the elements of N are either all multiples of $(1,1)^{\mathrm{T}}$ or all multiples of $(1,-1)^{\mathrm{T}}$). There are suitable lightlike vectors l_1, l_2 and a vector $d \in \mathbf{R}^4$ such that $W_1 = W[l_1, l_2, d]$. We have shown that either ($\overline{W_1} \cap \overline{W_2} \subset \overline{W_1} \backslash W_1$ is convex)

$$\emptyset \;=\; \overline{W[l_1, l_2, d + \lambda l_1]} \cap (\overline{W_1} \cap \overline{W_2}) \;=\; \overline{W[l_1, l_2, d + \lambda l_1]} \cap \overline{W_2} \quad \forall \lambda > 0$$

or

$$\emptyset \;=\; \overline{W[l_1, l_2, d - \lambda l_2]} \cap (\overline{W_1} \cap \overline{W_2}) \;=\; \overline{W[l_1, l_2, d - \lambda l_2]} \cap \overline{W_2} \quad \forall \lambda > 0.$$

We set $W(\lambda) = W[l_1, l_2, d + \lambda l_1]$ in the first case and $W(\lambda) = W[l_1, l_2, d - \lambda l_2]$ in the second case ($\lambda \in \mathbf{R}$). Such families of wedges were called characteristic in [18]. Note

$$\overline{W(\lambda)} \cap \overline{W_2} \;=\; \emptyset \quad \forall \lambda > 0. \tag{3.1.3}$$

There are suitable lightlike vectors k_1, k_2, a vector $d' \in \mathbf{R}^4$, and a function $f : \mathbf{R} \mapsto \mathbf{R}$ such that either

$$\tau(W(\lambda)) = W[k_1, k_2, d' + f(\lambda) k_1] \quad \text{or} \quad \tau(W(\lambda)) = W[k_1, k_2, d' - f(\lambda) k_2],$$

since characteristic families are mapped onto characteristic families for (3.1.2) is satisfied [18, Lemma 4.1.2]. f is bijective and monotone increasing according to (3.1.2), i.e., f is continuous. Then (if $\tau(W(\lambda)) = W[k_1, k_2, d' + f(\lambda)k_1]$)

$$\begin{aligned} \tau(W_1) \cap \tau(W_2) &= W[k_1, k_2, d' + f(0)k_1] \cap \tau(W_2) \\ &= (\cup_{\lambda>0} W[k_1, k_2, d' + f(\lambda)k_1]) \cap \tau(W_2) \\ &= \cup_{\lambda>0} (W[k_1, k_2, d' + f(\lambda)k_1] \cap \tau(W_2)) \\ &= \cup_{\lambda>0} (\tau(W(\lambda)) \cap \tau(W_2)) \\ &\overset{(3.1.1),(3.1.3)}{=} \emptyset \end{aligned}$$

The case $\tau(W(\lambda)) = W[k_1, k_2, d' - f(\lambda)k_2]$ follows similarly.

Lemma 3.1.2: *Assume that the strong CGMA is satisfied. The bijections on $\mathcal{W}$ induced by the adjoint action of modular conjugations of wedge algebras fulfill (A) and (B).*

Proof: The modular conjugations of wedge algebras are unitary and leave Ω fixed. For arbitrary $W_1, W_2 \in \mathcal{W}$, the set $(\mathcal{R}(W_1) \cap \mathcal{R}(W_2))\Omega$ is dense in $\mathcal{H}$ if and only if

$$J_W(\mathcal{R}(W_1) \cap \mathcal{R}(W_2))\Omega = ((J_W \mathcal{R}(W_1) J_W) \cap (J_W \mathcal{R}(W_2) J_W))\Omega$$

is dense in $\mathcal{H}$. Furthermore, $A\Omega \neq B\Omega$ for $A, B \in \mathcal{R}(W_1) \cap \mathcal{R}(W_2)$ if and only if

$$J_W A J_W \Omega = J_W A \Omega \neq J_W B \Omega = J_W B J_W \Omega.$$

(A) follows now from Lemma 3.1.1 and (ii), (iii) of the strong CGMA. (B) is evidently satisfied.

It will be proven that a map $\tau : \mathcal{W} \mapsto \mathcal{W}$ which satisfies (A) and (B) can be implemented by an element of the extended Poincaré group $\mathcal{DP}$. This result will follow from the known fact that if a bijection on $\mathbf{R}^4$ and its inverse preserve spacelike separation, then this map acts as an element of the extended Poincaré group [2].

Theorem 3.1.3: *Assume that* $\tau : \mathcal{W} \to \mathcal{W}$ *is bijective and fulfills (A) and (B). Then there exists an element* δ *of the extended Poincaré group* $\mathcal{DP}$ *such that*

$$\tau W = \{\delta(x) | x \in W\} \quad \text{for} \quad W \in \mathcal{W}.$$

If τ *is an involution, then* δ *is an element of the Poincaré group* $\mathcal{P}$.

The theorem will be proven in several steps. The conclusion in Theorem 3.1.3 was already obtained in an old version of [18] by assuming (A),

(C) $\tau(W') = \tau(W)'$, for all $W \in \mathcal{W}$,

and the following condition which is stronger than (B):

$$W_1, W_2, W \in \mathcal{W} \quad \text{satisfy} \quad W_1 \cap W_2 \subset W \quad \Leftrightarrow \quad \tau(W_1) \cap \tau(W_2) \subset \tau(W)$$

For each lightlike vector $l \neq 0$ and $c \in \mathbf{R}$, characteristic hyperplanes can be defined by

$$H_c[l] = \{x \in \mathbf{R}^4 \,|\, (x, l) = c\}.$$

A pair (W_1, W_2) of wedges with trivial intersection will be called maximal if there is no wedge W properly containing W_1, respectively W_2, such that $W \cap W_2 = \emptyset$, respectively $W_1 \cap W = \emptyset$. If a bijection $\tau : \mathcal{W} \mapsto \mathcal{W}$ fulfills (A) and (B), then the maximality of a pair of wedges is preserved under τ. This fact will play a fundamental role in the proof of Theorem 3.1.3.

Note the following result already obtained in [18, Lemma 4.1.2, Lemma 4.1.8] (characteristic families of wedges were defined in the proof of Lemma 3.1.1).

Lemma 3.1.4: *Let $\mathcal{F}$ be a characteristic family of wedges. If (B) is satisfied, then $\tau\mathcal{F}$ is again a characteristic family. The half-space $\cup_{W\in\mathcal{F}}\tau(W)$ depends only on the half-space $\cup_{W\in\mathcal{F}}W$ if (A), (B), and (C) are fulfilled. In particular, τ induces then a bijection on the set of characteristic hyperplanes, which we also call τ.*

In order to be able to apply Lemma 3.1.4 we have to show that (C) is a consequence of (A) and (B) (Corollary 3.1.8).

We need a characterization for the maximality of a pair of wedges. Given a pair (W_1, W_2) such that we cannot obtain W_2 from W_1 or W_1' by a translation, i.e., such that W_2 is neither coherent to W_1 nor to W_1', there is a Poincaré transformation L mapping W_1 onto W_R and W_2 onto a wedge $W[l_{2+}, (1,a,b,c)^{\mathrm{T}}, d]$, resp. $W[l_{2+}, (1,a,b,c)^{\mathrm{T}}, d]'$, for suitable $a, b, c \in \mathbf{R}$ with $a^2+b^2+c^2 = 1$, $b \neq 1$, and $d \in \mathbf{R}^4$. Therefore, we will consider the pair (LW_1, LW_2) and determine under which conditions this pair is maximal (Lemma 3.1.6).

Lemma 3.1.5: *Let $P : \mathbf{R}^4 \mapsto \mathbf{R}^2$ be the projection onto the first two components and $W_1 = W[l_{2+}, (1,a,b,c)^{\mathrm{T}}, d]$ $(a,b,c \in \mathbf{R},\ d \in \mathbf{R}^4,\ b \neq 1,\ a^2+b^2+c^2 = 1)$. Then $PW_1 = \mathbf{R}^2$ for $b < 0$ or $c \neq 0$. In the other case, i.e., if $0 \leq b < 1$ and $c = 0$, we obtain*

$$PW_1 = \{x \in \mathbf{R}^2 | (x - Pd)^{\mathrm{T}} \begin{pmatrix} 1-b \\ -a \end{pmatrix} > 0\}.$$

Proof: Without loss of generality we can assume $d = 0$. We know $(1 - b \neq 0)$

$$\begin{aligned} PW_1 &= P\{\alpha l_{2+} + \beta(1,a,b,c)^{\mathrm{T}} + s(c,0,c,1-b)^{\mathrm{T}} \\ &\quad + t(a,1-b,a,0)^{\mathrm{T}} | \alpha > 0, \beta < 0,\ s,t \in \mathbf{R}\} \\ &= \{\alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}1\\a\end{pmatrix} + s\begin{pmatrix}c\\0\end{pmatrix} + t\begin{pmatrix}a\\1-b\end{pmatrix} | \alpha > 0, \beta < 0, s,t \in \mathbf{R}\}. \end{aligned}$$

This shows $PW_1 = \mathbf{R}^2$ for $c \neq 0$ $(1 - b \neq 0)$, i.e., we can restrict ourselves to $c = 0$. $\binom{1-b}{-a}$ is a normal vector of the line $\{t\binom{a}{1-b} | t \in \mathbf{R}\}$. The claim follows now easily with

$$(\alpha(1,0))\begin{pmatrix}1-b\\-a\end{pmatrix} = \alpha(1-b) > 0, \quad \text{for} \quad \alpha > 0,$$

and $(c = 0)$

$$(\beta(1,a))\begin{pmatrix}1-b\\-a\end{pmatrix} = \beta(1-b-a^2) = \beta(b^2-b), \quad \beta < 0,$$

which is nonnegative for $0 \leq b < 1$ and negative for $b < 0$.

Lemma 3.1.6: *The wedges* $W_R = W[l_{1+}, l_{1-}, 0]$ *and* $W_1 = W[l_{2+}, (1,a,b,c)^{\mathrm{T}}, d]$ $(a,b,c \in \mathbf{R},\ d \in \mathbf{R}^4,\ b \neq 1,\ a^2 + b^2 + c^2 = 1)$ *are maximal with trivial intersection if and only if* $0 < a < 1$, $0 < b < 1$, $c = 0$, *and* $d \equiv 0$ *modulo vectors whose associated translations leave* W_R *or* W_1 *fixed.*

The statement is still true if we replace W_1 *by* W_1' *and the condition* $0 < a < 1$ *by* $-1 < a < 0$ *or also if we replace* l_{2+} *by* l_{2-} *and* $0 < b < 1$ *by* $-1 < b < 0$.

Proof: Note that $x \in PW_R$ (see Lemma 3.1.5 for the definition of P) if and only if

$$x = \alpha\begin{pmatrix}1\\1\end{pmatrix} - \beta\begin{pmatrix}1\\-1\end{pmatrix} \quad \text{for suitable} \quad \alpha, \beta > 0. \tag{3.1.4}$$

W_R is invariant with respect to translations by vectors which are multiples of $(0,0,1,0)^{\mathrm{T}}$ and $(0,0,0,1)^{\mathrm{T}}$, i.e., $W_R \cap W_1 = \emptyset$ if and only if $PW_R \cap PW_1 = \emptyset$. According to

Lemma 3.1.5, $PW_R \cap PW_1 = \emptyset$ is equivalent to $c = 0$, $0 \leq b < 1$ (PW_1 is a half-plane) and (use (3.1.4))

$$\begin{aligned}((\alpha\begin{pmatrix}1\\1\end{pmatrix} - \beta\begin{pmatrix}1\\-1\end{pmatrix}) - Pd)^{\mathrm{T}}\begin{pmatrix}1-b\\-a\end{pmatrix} &= \alpha(1,1)\begin{pmatrix}1-b\\-a\end{pmatrix} - \beta(1,-1)\begin{pmatrix}1-b\\-a\end{pmatrix} \\ &\quad -(Pd)^{\mathrm{T}}\begin{pmatrix}1-b\\-a\end{pmatrix} \\ &\leq 0 \quad \forall \alpha, \beta > 0, \end{aligned} \tag{3.1.5}$$

i.e., to $c = 0$, $0 \leq b < 1$, $-(Pd)^{\mathrm{T}}\binom{1-b}{-a} \leq 0$,

$$1 - b - a = (1,1)\begin{pmatrix}1-b\\-a\end{pmatrix} \leq 0 \tag{3.1.6}$$

and

$$-1 + b - a = -(1,-1)\begin{pmatrix}1-b\\-a\end{pmatrix} \leq 0. \tag{3.1.7}$$

Therefore, $W_R \cap W_1 = \emptyset$ is equivalent to $c = 0$, $0 \leq b < 1$, $-(Pd)^{\mathrm{T}}\binom{1-b}{-a} \leq 0$, and $a, b \geq 0$. Note that $a, b \geq 0$ and $c = 0$ imply $a > 0$ ($b \neq 1$).

Assume first the maximality of (W_R, W_1). Then $-(Pd)^{\mathrm{T}}\binom{1-b}{-a} \leq 0$ and (3.1.4) - (3.1.7) entail

$$(x - Pd)^{\mathrm{T}}\begin{pmatrix}1-b\\-a\end{pmatrix} \leq -(Pd)^{\mathrm{T}}\begin{pmatrix}1-b\\-a\end{pmatrix} \leq 0, \tag{3.1.8}$$

for $x \in PW_R$. The maximality implies

$$-(Pd)^{\mathrm{T}}\begin{pmatrix}1-b\\-a\end{pmatrix} = 0 \tag{3.1.9}$$

(if not, we could obtain a wedge which properly contains W_1 and also possesses trivial intersection with W_R by choosing a different d such that (3.1.8) is still fulfilled), i.e., Pd is a multiple of $\binom{a}{1-b} = P(a, 1-b, a, 0)^{\mathrm{T}}$. Therefore, d is a linear combination of $(a, 1-b, a, 0)^{\mathrm{T}}$, $(0,0,1,0)^{\mathrm{T}}$, and $(0,0,0,1)^{\mathrm{T}}$ and d is of the stated form.

We still have to exclude $b = 0$. $b = 0$ would entail $a = 1$ and that W_1, respectively PW_1, is invariant with respect to translations by multiples of $(1,1,1,0)^{\mathrm{T}}$, respectively $P(1,1,1,0)^{\mathrm{T}} = \binom{1}{1}$. If we translate the pair (W_1, W_R) by $-(1,1,1,0)^{\mathrm{T}}$, we get a pair (W_1, W_2) with trivial intersection such that $W_2 = W[l_{1+}, l_{1-}, (-1,-1,0,0)^{\mathrm{T}}]$ properly contains W_R. This contradicts the maximality of (W_1, W_R).

Assume now that W_1 is of the stated form. We already know $W_1 \cap W_R = \emptyset$ (see above). It remains to prove the maximality. According to the assumptions, (3.1.9) is true. Furthermore, (3.1.6) and (3.1.7) are fulfilled with $\leq$ replaced by $<$. Every wedge W_3 which contains W_R is obtained from W_R by a translation by a vector $-\alpha_0 l_{1+} + \beta_0 l_{1-}$, $\alpha_0, \beta_0 \geq 0$. For $W_3 \neq W_R$, i.e., for $\alpha_0 \neq 0$ or $\beta_0 \neq 0$, (3.1.6) and (3.1.7) ($\leq$ replaced by $<$) imply

$$(P(-\alpha_0 l_{1+} + \beta_0 l_{1-}))^{\mathrm{T}} \begin{pmatrix} 1-b \\ -a \end{pmatrix} > 0.$$

The vertex of PW_3 lies in PW_1 (Lemma 3.1.5 and (3.1.9)), i.e., $PW_3 \cap PW_1 \neq \emptyset$, respectively $W_3 \cap W_1 \neq \emptyset$. Similarly, it follows that we cannot replace W_1 by a wedge containing W_1 and still having trivial intersection with W_R.

In order to prove the last statement consider the transformed wedges under suitable reflections.

We can also give a coordinate independent characterization of maximal pairs of wedges. Since we will not use the following remark, we refrain from giving the proof.

Remark: A pair of wedges $(W[l_1, l_2, d_1], W[l_3, l_4, d_2])$ ($d_1, d_2 \in \mathbf{R}^4$, l_1, l_2, resp. l_3, l_4, linearly independent future-directed lightlike vectors in $\mathbf{R}^4$) is maximal iff

$$\lambda l_1 + \mu l_3 = \lambda' l_2 + \mu' l_4,$$

where $\lambda, \lambda', \mu, \mu' > 0$ and $d_1 \equiv d_2$ modulo vectors whose associated translations leave $W[l_1, l_2, d_1]$ or $W[l_3, l_4, d_2]$ fixed.

Lemma 3.1.7: *Let* $W_1, W_2 \in \mathcal{W}$ *be arbitrary.* $W_2 \subset W_1'$ *if and only if the two standard families* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *containing* W_2 *fulfill* $W_1 \cap W = \emptyset \ \forall W \in \mathcal{F}_1 \cup \mathcal{F}_2$.

Proof: see [18, Lemma 4.1.4]

Corollary 3.1.8: *A bijection* $\tau : \mathcal{W} \mapsto \mathcal{W}$ *which fulfills (A) and (B) also fulfills (C).*

Proof: Lemma 3.1.4 and Lemma 3.1.7 entail

$$\tau(W') \subset \tau(W)' \quad \forall W \in \mathcal{W}.$$

The claim follows by using the same relation for τ^{-1}.

For each Lorentz transformation L we will call the induced bijection on $\mathcal{W}$, respectively on the set of characteristic hyperplanes, also L.

Lemma 3.1.9: *Let* $\tau : \mathcal{W} \mapsto \mathcal{W}$ *be a bijection that fulfills (A) and (B), i.e.,* τ *induces a bijection on the set of characteristic hyperplanes, and let* l_1, l_2, l_3, l_4 *be lightlike vectors which are linearly dependent but pairwise linearly independent. Then*

$$\cap_{i=1}^{4} \tau H_0[l_i] \; = \; \cap_{i \neq k} \tau H_0[l_i], \quad \text{for} \quad k = 1,2,3,4.$$

Proof: Note that for a maximal pair of wedges $(W[k_1, k_2, d_1], W[k_3, k_4, d_2])$ ($d_1, d_2 \in \mathbf{R}^4$, k_1, k_2, resp. k_3, k_4, linearly independent in the forward lightcone) and suitable $c_1, c_2, c_3, c_4 \in \mathbf{R}$ which are chosen such that $H_{c_1}[k_1]$ and $H_{c_2}[k_2]$, resp. $H_{c_3}[k_3]$ and $H_{c_4}[k_4]$, are the characteristic half-planes determining $W[k_1, k_2, d_1]$, resp. $W[k_3, k_4, d_2]$,

$$\cap_{i=1}^{4} H_{c_i}[k_i] \; = \; \cap_{i \neq j} H_{c_i}[k_i], \quad \text{for} \quad j = 1,2,3,4 \tag{3.1.10}$$

(see the remark preceding Lemma 3.1.5; Lemma 3.1.6). By considering τL for a suitable Lorentz transformation L we can assume $l_1 = l_{1+}$, $l_2 = l_{1-}$, $l_3 = l_{2+}$ and $l_4 = (1, a, b, 0)^{\mathrm{T}}$, for suitable $a, b \in \mathbf{R}$ with $a^2 + b^2 = 1$. If the pair $(W_R, W[l_{2+}, l_4, 0])$, resp. $(W_R, W[l_{2+}, l_4, 0]')$, is maximal, the claim follows from (3.1.10), i.e., without loss of generality $b \leq 0$ (Lemma 3.1.6). $b = 0$ is excluded, since l_1, l_2, l_3, l_4 are pairwise linearly independent. Set $l = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)^{\mathrm{T}}$. Using (3.1.10) and the maximal pairs $(W_R, W[l_{2+}, l, 0])$ and $(W[l_{2+}, l_{2-}, 0], W[l_{1+}, l, 0])$ we conclude

$$\begin{aligned}
\cap_{i=1}^{3} \tau H_0[l_i] &= \tau H_0[l_{1+}] \cap \tau H_0[l_{1-}] \cap \tau H_0[l_{2+}] \\
&= \tau H_0[l_{1+}] \cap \tau H_0[l_{1-}] \cap \tau H_0[l_{2+}] \cap \tau H_0[l] \\
&= \tau H_0[l_{1+}] \cap \tau H_0[l] \cap \tau H_0[l_{2+}] \\
&= \tau H_0[l_{1+}] \cap \tau H_0[l] \cap \tau H_0[l_{2+}] \cap \tau H_0[l_{2-}] \\
&\subset \tau H_0[l_{2-}],
\end{aligned}$$

i.e. (if l_4 is not a multiple of l_{2-}, $(W_R, W[l_{2-}, l_4, 0])$, resp. $(W_R, W[l_{2-}, l_4, 0]')$, is maximal because of $b < 0$ and Lemma 3.1.6),

$$\begin{aligned}
\cap_{i=1}^{3} \tau H_0[l_i] &\subset \tau H_0[l_{1+}] \cap \tau H_0[l_{1-}] \cap \tau H_0[l_{2-}] \\
&= \tau H_0[l_{1+}] \cap \tau H_0[l_{1-}] \cap \tau H_0[l_{2-}] \cap \tau H_0[l_4] \\
&\subset \tau H_0[l_4].
\end{aligned}$$

Therefore,

$$\cap_{i=1}^{4} \tau H_0[l_i] = \cap_{i=1}^{3} \tau H_0[l_i].$$

Lemma 3.1.10: *If $\tau : \mathcal{W} \mapsto \mathcal{W}$ is bijective and fulfills (A) and (B), then $\cap_l \tau H_0[l]$ ($l \neq 0$ are the lightlike vectors) is a singleton set.*

Proof: There are lightlike vectors k_1, k_2, k_3, k_4 and $c_1, c_2, c_3, c_4 \in \mathbf{R}$ such that $\tau H_0[l_{1+}] = H_{c_1}[k_1]$, $\tau H_0[l_{1-}] = H_{c_2}[k_2]$, $\tau H_0[l_{2+}] = H_{c_3}[k_3]$ and $\tau H_0[l_{3+}] = H_{c_4}[k_4]$. According to Lemma 3.1.4, τ, resp. τ^{-1}, maps parallel characteristic hyperplanes onto parallel characteristic hyperplanes (use (B)). Hence, there are $b_1, b_2, b_3, b_4 \in \mathbf{R}$ such that $\tau H_{b_1}[l_{1+}] = H_0[k_1]$, $\tau H_{b_2}[l_{1-}] = H_0[k_2]$, $\tau H_{b_3}[l_{2+}] = H_0[k_3]$, and $\tau H_{b_4}[l_{3+}] = H_0[k_4]$. If k_1, k_2, k_3, k_4 would be linearly dependent, then also $l_{1+}, l_{1-}, l_{2+}, l_{3+}$ (Lemma 3.1.9 for τ^{-1}). Thus, k_1, k_2, k_3, k_4 are linearly independent, i.e., $\cap_{i=1}^4 H_{c_i}[k_i]$ is a singleton set. An arbitrary lightlike vector $k \neq 0$ is a linear combination of l_{3+} and two lightlike vectors l_1, l_2 with trivial x_3-component. According to Lemma 3.1.9 (if l_1, l_2, l_{3+}, k are not pairwise linearly independent, then k is a multiple of l_1, l_2 or l_{3+}),

$$\tau H_0[l_1] \cap \tau H_0[l_2] \cap \tau H_0[l_{3+}] \ \subset \tau H_0[k]$$

and also

$$\tau H_0[l_{1+}] \cap \tau H_0[l_{1-}] \cap \tau H_0[l_{2+}] \ \subset \tau H_0[l_j], \quad \text{for} \quad j = 1, 2.$$

This proves the claim because of

$$\cap_{i=1}^4 H_{c_i}[k_i] \ = \ \tau H_0[l_{1+}] \cap \tau H_0[l_{1-}] \cap \tau H_0[l_{2+}] \cap \tau H_0[l_{3+}] \ \subset \ \tau H_0[k],$$

for arbitrary lightlike $k \neq 0$.

Proof of Theorem 3.1.3: Let T_x be the translation by x on $\mathbf{R}^4$ for arbitrary $x \in \mathbf{R}^4$. Define $\delta : \mathbf{R}^4 \mapsto \mathbf{R}^4$ by

$$\{\delta(x)\} \ = \bigcap_{l \neq 0 \text{ lightlike}} \tau T_x H_0[l], \quad \text{for} \quad x \in \mathbf{R}^4.$$

δ is well-defined (Lemma 3.1.10 for τT_x). To arbitrary $W_0 \in \mathcal{W}$ and $y \in W_0$ there is a wedge $W_1 \subset W_0$ such that y is in the edge of W_1. Furthermore, the characteristic hyperplanes defining W_0 are different from the hyperplanes defining W_1. The same must be true for τW_0 and τW_1, i.e., $\overline{\tau W_1} \subset \tau W_0$. Let H_1, H_2 be the characteristic hyperplanes defining W_1. There are two characteristic families $\mathcal{F}_1$ and $\mathcal{F}_2$ which contain W_1 such that $H_1 = \partial(\cup_{W \in \mathcal{F}_1} W)$, resp. $H_2 = \partial(\cup_{W \in \mathcal{F}_2} W)$. τW_1 is contained in $\tau\mathcal{F}_1$ and $\tau\mathcal{F}_2$, i.e., $\tau H_1 = \partial(\cup_{W \in \tau\mathcal{F}_1} W)$ and $\tau H_2 = \partial(\cup_{W \in \tau\mathcal{F}_2} W)$ are the characteristic hyperplanes defining τW_1, since there are exactly two characteristic families ($\tau\mathcal{F}_1$ and $\tau\mathcal{F}_2$) containing τW_1. The characteristic hyperplanes containing y (H_1 and H_2 belong to this set) are mapped into the set of characteristic hyperplanes containing $\delta(y)$, i.e., $\delta(y) \in \tau H_1$ and $\delta(y) \in \tau H_2$. This shows that $\delta(y)$ lies in the two characteristic hyperplanes defining τW_1. We can conclude $\delta(y) \in \overline{\tau W_1} \subset \tau W_0$, i.e.,

$$\tau W \supset \{\delta(x) | x \in W\} \quad \text{for} \quad \text{every} \quad W \in \mathcal{W}. \tag{3.1.11}$$

We obtain the other inclusion by considering τ^{-1}.

$$\bigcap_{l \neq 0 \text{ lightlike}} \tau^{-1} T_z H_0[l], \quad z \in \tau W \quad \text{arbitrary},$$

is a singleton set in $\tau^{-1}(\tau W) = W$ (use (3.1.11) for τ^{-1}), i.e., in particular,

$$\{\tau^{-1} T_z H_0[l] \,|\, l \neq 0 \text{ lightlike}\} \;=\; \{T_w H_0[l] \,|\, l \neq 0 \text{ lightlike}\},$$

for a suitable $w \in W$. $\delta(w)$ is contained in $\tau T_w H_0[l]$ for arbitrary lightlike $l \neq 0$, i.e.,

$$\delta(w) \in \tau\tau^{-1} T_z H_0[l] \;=\; T_z H_0[l],$$

for arbitrary lightlike $l \neq 0$. This implies $\delta(w) = z$ and $\{\delta(x) | x \in W\} = \tau W$. If x, y are spacelike separated, then there are wedges $W_1, W_2 \in \mathcal{W}$ such that $x \in W_1$, $y \in W_2$, and $W_1 \subset W_2'$. Hence, τ and τ^{-1} are bijections preserving spacelike separation and we can conclude that τ acts as an element of the extended Poincaré group [2].

3.2 Which Transformation Groups Can Appear?

In the following we will assume that the modular conjugations J_W of the pairs $(\mathcal{R}(W),\Omega)$ act as Poincaré transformations g_W and that $W \mapsto \mathcal{R}(W)$ is an order-preserving bijection. This can be derived from the strong CGMA.

Denote the group generated by $\{g_W \,|\, W \in \mathcal{W}\}$ by $\mathcal{G}$. We want to find the possible groups $\mathcal{G}$. We will prove that $\mathcal{G} = \mathcal{P}_+$, $\mathcal{G} \cong \mathrm{IO}(3)$, $\mathcal{G} \cong \mathrm{IO}(3) \times \mathbf{Z}_2$, $\mathcal{G} \cong \mathrm{IO}(3) \times \mathrm{IO}(1)$ or $\mathcal{G}$ consists of the transformations whose homogeneous part leaves $\{\pm l\}$, for a lightlike vector $l \neq 0$, invariant (Theorem 3.2.2.8). Under the additional assumption of wedge duality, we will determine in Corollary 3.2.2.9 the geometric action of the modular conjugations, which is, in each case, up to the choice of the coordinate system, uniquely determined (only in the case $\mathcal{G} \cong \mathrm{IO}(3) \times \mathbf{Z}_2$, there are, for certain wedges $W \in \mathcal{W}$, two ways how g_W can act).

We can introduce a topology on $\mathcal{W}$ [18, 4.3]. $\mathcal{W} = \mathcal{P}_+^\uparrow W_R$ is in 1-1 correspondence with the quotient space $\mathcal{P}_+^\uparrow/\mathrm{Inv}\mathcal{P}_+^\uparrow(W_R)$ (we denote by $\mathrm{Inv}\mathcal{P}_+^\uparrow(W_R)$ the subgroup of $\mathcal{P}_+^\uparrow$ consisting of the elements which leave W_R invariant). The quotient topology on $\mathcal{P}_+^\uparrow/\mathrm{Inv}\mathcal{P}_+^\uparrow(W_R)$ thereby induces a topology on $\mathcal{W}$. In [18, Lemma 4.3.2] we assumed a net continuity assumption which entails the continuity of $W \mapsto J_W$.

If we assume wedge duality and the continuity of $W \mapsto J_W$, then it can be shown that either $\mathcal{G} = \mathcal{P}_+$ or $\mathcal{G} \cong \mathrm{IO}(3)$ (Corollary 3.2.2.10).

Frequently, we will use in the following $g_{W_1} g_{W_2} W_2 \subset W_1$ for wedges satisfying $W_1 \subset W_2$. This is a consequence of

$$\begin{aligned}
\mathcal{R}(g_{W_1} g_{W_2} W_2) &= J_{W_1} J_{W_2} \mathcal{R}(W_2) J_{W_2} J_{W_1} \\
&= J_{W_1} \mathcal{R}(W_2)' J_{W_1} \\
&\subset J_{W_1} \mathcal{R}(W_1)' J_{W_1} \\
&= \mathcal{R}(W_1). \qquad (3.2.1)
\end{aligned}$$

Similarly, we can prove $g_{W_1} g_{W_2} W_2 \supset W_1$ if $W_1 \supset W_2$. Furthermore, it will be used that g_W commutes with the elements of $\mathcal{G}$ which leave this wedge invariant (uniqueness of the modular objects). Note the following notations:

$$W_k^{(0)} \equiv \{x \in \mathbf{R}^4 \,|\, x_k > |x_0|\}, \; k = 2,3$$

Let $R_i(\alpha)$ be the rotation by α about the x_i-axis.

3.2.1 Two-Dimensional Minkowski Space

First we have to solve the corresponding problem in $1+1$ dimensions. Let $\mathcal{W}_1$ be the set of wedges in $1+1$ dimensions. We assume that the modular conjugations J_W of wedge algebras act as Poincaré transformations g_W and denote the group generated by $\{g_W \,|\, W \in \mathcal{W}_1\}$ by $\mathcal{G}_1$.

Lemma 3.2.1.1: *g_W, $W \in \mathcal{W}_1$, is of the following form*

$$g_W x = B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + B \begin{pmatrix} 0 \\ c \end{pmatrix} \quad \forall x \in \mathbf{R}^2,$$

where B is a boost and $c \in \mathbf{R}$, or

$$g_W x = -x + a \quad \forall x \in \mathbf{R}^2,$$

where $a \in \mathbf{R}^2$.

Proof: The Lorentz group consists of elements of the form $B_1 \operatorname{diag}(\pm 1, \pm 1)$, where B_1 is a boost. The claim follows from $g_W^2 = 1$. We only have to exclude

$$g_W x = B \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} x + B \begin{pmatrix} c \\ 0 \end{pmatrix} \quad \forall x \in \mathbf{R}^2$$

($B = 1$ and $c = 0$ for $+$). However, this would imply $g_W W \cap W \in \mathcal{W}_1$ and that

$$\mathcal{R}(g_W W \cap W) \subset \mathcal{R}(g_W W) \cap \mathcal{R}(W) = \mathcal{R}(W)' \cap \mathcal{R}(W)$$

is abelian. This is impossible, since an abelian von Neumann algebra with cyclic vector is maximally abelian.

Lemma 3.2.1.2: *Let* $\{W_n \in \mathcal{W}_1\}$ *and* $\{V_n \in \mathcal{W}_1\}$ *be increasing sequences of wedges such that, for a* $W \in \mathcal{W}_1$, $\overline{W_n}, \overline{V_n} \subset W$, $W = \cup_n W_n = \cup_n V_n$, *and* $g_{W_n} \to g_W$. *This implies* $g_{V_n} \to g_W$.

If $\{W_n \in \mathcal{W}_1\}$ *and* $\{V_n \in \mathcal{W}_1\}$ *are decreasing sequences of wedges such that* $\overline{W} \subset W_n$, $\overline{W} \subset V_n$, $\overline{W} = \cap_n W_n = \cap_n V_n$, *and* $g_{W_n} \to g_W$, *then it follows* $g_{V_n} \to g_W$.

Proof: First we consider increasing sequences. Note that $V_n \subset W_m$, resp. $W_n \subset V_m$, for fixed n and sufficiently large m. This entails $\cup_n \mathcal{R}(W_n) = \cup_n \mathcal{R}(V_n)$. Hence, $\{J_{W_n}\}$ and $\{J_{V_n}\}$ converge strongly to the modular conjugation of $((\cup_n \mathcal{R}(W_n))'' = (\cup_n \mathcal{R}(V_n))'', \Omega)$ [17, Corollary A.2]. We will use this fact frequently in the following proof.

We can assume $W = W_R$. If we replace $\{W_n\}$ by a suitable subsequence, we can assume

$$g_{V_n} g_{W_n} W_R \ \subset \ W_R, \tag{3.2.1.1}$$

since

$$\overline{g_{V_n} g_{W_R} W_R} \ \subset \ \overline{V_n} \ \subset \ W_R$$

and $\lim_n g_{W_n} \to g_{W_R}$. Furthermore, according to Lemma 3.2.1.1, we can assume

$$g_{V_n} g_{W_n} x \ = \ B_n \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} x + a_n$$

with boosts B_n and $a_n \in \mathbf{R}^2$ satisfying $W_R + a_n \subset W_R$ (see (3.2.1.1)). Note that there cannot be an $a \neq 0$ such that for infinitely many n

$$W_R \subset W_R + a \subset g_{V_n} g_{W_n} W_R, \tag{3.2.1.2}$$

since the strong convergence of $J_{V_n} J_{W_n} \to 1$ would imply, for a suitable subsequence $\{m_n\}$,

$$A \;=\; \lim_n J_{V_{m_n}} J_{W_{m_n}} A J_{W_{m_n}} J_{V_{m_n}} \;\in\; \mathcal{R}(g_{V_{m_n}} g_{W_{m_n}} W_R) \;\subset\; \mathcal{R}(W_R + a)$$

$\forall A \in \mathcal{R}(W_R)$. It is not difficult to conclude $a_n \to 0$ (otherwise, there is an $a \neq 0$ satisfying (3.2.1.2) for infinitely many n). Similarly, we will show $B_n \to 1$. Assume that $\{B_n\}$ does not converge to 1. There are a lightlike vector $l \neq 0$ and a $\lambda \in \mathbf{R}$ such that $W_R + l \subset W_R$ and $B_{k_n} l = \lambda_{k_n} l$ with $\lambda_{k_n} > \lambda > 1$, for a suitable subsequence $\{k_n\}$.

$$J_{V_{k_n}} J_{W_{k_n}} \mathcal{R}(W_R + \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} l) J_{W_{k_n}} J_{V_{k_n}} \;=\; \mathcal{R}(W_R + \lambda_{k_n} l + a_{k_n}) \;\subset\; \mathcal{R}(W_R + \lambda l)$$

gives the contradiction

$$\mathcal{R}(W_R + \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} l) \;\subset\; \mathcal{R}(W_R + \lambda l).$$

Similar arguments prove $g_{V_n} g_{W_n} x = B_n x + a_n$ for all but finitely many n. Hence, we have shown $g_{V_n} \to \lim_n g_{W_n} = g_W$.

We can proceed similarly if we consider decreasing sequences. We can use that $\{J_{W_n}\}$ and $\{J_{V_n}\}$ converge strongly to the modular conjugation of $\cap_n \mathcal{R}(W_n) = \cap_n \mathcal{R}(V_n) \supset \mathcal{R}(W)$ and we can assume, instead of (3.2.1.1), that

$$g_{W_n} g_{V_n} W_R \subset W_R,$$

since

$$\overline{g_{W_R} g_{V_n} W_R} \;=\; g_{W_R} g_{V_n} \overline{W_R} \;\subset\; g_{W_R} g_{V_n} V_n \;\subset\; W_R$$

and $\lim_n g_{W_n} \to g_{W_R}$ (replace $\{W_n\}$ by a suitable subsequence). The remainder of the proof is identical (consider $g_{W_n} g_{V_n}$ instead of $g_{V_n} g_{W_n}$).

Lemma 3.2.1.3: *Let $B \neq 1$ be a boost. If T is a spacelike (timelike) translation, then $[B,T] = B^{-1}T^{-1}BT$ is timelike (spacelike). Furthermore, to any map of the form $x \mapsto Bx + c$, $c \in \mathbf{R}^2$, there is a translation T such that $Bx + c = T^{-1}BTx$ $\forall x \in \mathbf{R}^2$.*

Proof: There are future-directed lightlike vectors l_1, l_2 such that $Bl_k = \lambda_k l_k$ for suitable $\lambda_1 > 1$, $\lambda_2 < 1$. Note that $c_1 l_1 + c_2 l_2$, $c_1, c_2 \in \mathbf{R}$, is spacelike (timelike) if and only if $c_1 c_2 < 0$ $(c_1 c_2 > 0)$. If T translates by a spacelike vector l, then there are $a, b \in \mathbf{R}$ such that $l = al_1 + bl_2$ and $ab < 0$. $[B,T] = B^{-1}T^{-1}BT$ is a translation by $l - B^{-1}l = a(1-\lambda_1^{-1})l_1 + b(1-\lambda_2^{-1})l_2$, which is timelike. Similarly, we can treat the case of timelike translations T.

It remains to prove the last claim. Let $a, b \in \mathbf{R}$ and T be the translation by $al_1 + bl_2$. We can choose a and b such that $T^{-1}BTx = Bx + a(\lambda_1 - 1)l_1 + b(\lambda_2 - 1)l_2 = Bx + c$ $(\lambda_k \neq 1)$.

Lemma 3.2.1.4: *$\mathcal{G}_1$ contains a sequence of spacelike translations converging towards 1.*

Proof: Set $W_c = W_R + c\binom{0}{1}$ for $c \in \mathbf{R}$. By considering the uncountable set $\{W_c \,|\, c \in \mathbf{R}\}$, we can derive the existence of a sequence $\{c_n \in \mathbf{R}\}$ such that the c_n are pairwise different and $\{g_{W_{c_n}}\}$ converges (discrete subsets of the symmetry group are countable).

Note that $(g_{W_{c_n}} g_{W_{c_m}})^2$ are nontrivial for $n \neq m$, since $g_{W_{c_n}} g_{W_{c_m}} W_{c_m}$ is contained in $W_{c_n} \subset W_{c_m}$ for $c_m < c_n$, resp. contains $W_{c_n} \supset W_{c_m}$ for $c_n < c_m$. $(g_{W_{c_n}} g_{W_{c_{n+1}}})^2$ provide either translations by arbitrarily small vectors, which are spacelike $((g_{W_{c_n}} g_{W_{c_{n+1}}})^2 W_{c_{n+1}} \subset W_{c_n} \subset W_{c_{n+1}}$ for $c_{n+1} < c_n$, resp. $(g_{W_{c_n}} g_{W_{c_{n+1}}})^2 W_{c_{n+1}} \supset$

$W_{c_n} \supset W_{c_{n+1}}$ for $c_n < c_{n+1}$), or, by considering a suitable subsequence, we can assume that they are of the form $x \mapsto B_n x + b_n$ (Lemma 3.2.1.1), where the boosts $\{B_n \neq 1\}$ converge to 1 and $\{b_n\}$ converges to 0. We only have to consider the latter case. If we choose the coordinate system suitably, we can assume that $\mathcal{G}_1$ contains a boost $B \neq 1$ (Lemma 3.2.1.3). This implies $g_{W_R} x = -x \; \forall x \in \mathbf{R}^2$, since this boost commutes with g_{W_R} (use Lemma 3.2.1.1). Hence, $(g_{W_R} g_{W_1})^2$ is a translation (Lemma 3.2.1.1), which is spacelike ($(g_{W_R} g_{W_1})^2 W_1 \supset W_R \supset W_1$), and

$$[(g_{W_R} g_{W_1})^2, (g_{W_{c_n}} g_{W_{c_{n+1}}})^2] \;=\; [(g_{W_R} g_{W_1})^2, B_n]$$

are timelike ($B_n \neq 1$, Lemma 3.2.1.3) and converge to 1 ($(g_{W_{c_n}} g_{W_{c_{n+1}}})^2 \to 1$). The claim follows now from the fact that

$$[[(g_{W_R} g_{W_1})^2, B_n], B_n] \;=\; [[(g_{W_R} g_{W_1})^2, (g_{W_{c_n}} g_{W_{c_{n+1}}})^2], (g_{W_{c_n}} g_{W_{c_{n+1}}})^2] \;\in\; \mathcal{G}_1$$

are spacelike (Lemma 3.2.1.3) and converge to 1.

Lemma 3.2.1.5: *The homogeneous part of g_W is independent of $W \in \mathcal{W}_1$.*

Proof: Let W_0 be a wedge such that g_{W_0} does not act as an inversion (if such a wedge does not exist, then there is nothing to prove). g_{W_0} is of the form

$$g_{W_0} x \;=\; B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + a,$$

where B is a boost and $a \in \mathbf{R}^2$ (Lemma 3.2.1.1). If $T \in \mathcal{G}_1$ is a spacelike translation, then $T^{-1} g_{W_0} T g_{W_0} \in \mathcal{G}_1$ is a translation by a nontrivial vector (g_{W_0} commutes only with translations in the timelike direction $B\binom{1}{0}$), which is inverted by $B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. These transformations translate by spacelike vectors, which can be arbitrarily small for appropriate choices of $T \in \mathcal{G}_1$ (Lemma 3.2.1.4). Let $v \neq 0$ be a vector which is inverted by $B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. v is spacelike, i.e. we can assume $W_0 + v \subset W_0$. There are nonzero $\{s_n\}$, which decrease to 0, such that the translations by $s_n v$ are elements of $\mathcal{G}_1$. Note the convergence of $g_{W_0+s_n v} \to g_{W_0}$ ($g_{W_0+s_n v}$ can be obtained from g_{W_0} by adjoining with the translation by $s_n v$, which is contained in $\mathcal{G}_1$). If $\{t_n\}$ is a sequence of real numbers decreasing to 0, then we can conclude $g_{W_0+t_n v} \to g_{W_0}$, according to Lemma 3.2.1.2. Similarly, we can proceed if $\{t_n\}$ increases to 0. Every sequence which converges to 0 has a decreasing or increasing subsequence $\{t_n\}$, i.e a subsequence such that $g_{W_0+t_n v} \to g_{W_0}$. This entails the continuity of $t \mapsto g_{W_0+tv}$ at $t = 0$ and, similarly, at any $t \in \mathbf{R}$.

Let T_t be the translation by tv. T_t is an element of $\mathcal{G}_1$ for t in a dense subset A of $\mathbf{R}$ and $g_{W_0} T_t g_{W_0} = T_{-t}$. The continuity of $t \mapsto g_{W_0+tv}$ implies

$$g_{W_0} g_{W_0+tv} = g_{W_0} T_t g_{W_0} T_{-t} = T_{-2t} \quad \forall t \in \mathbf{R},$$

since this is true for $t \in A$. Therefore, $\mathcal{G}_1$ contains all translations by multiples of v.

If there would be wedges $W_1, W_2 \in \mathcal{W}_1$ such that

$$g_{W_k} x = B_k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + B_k \begin{pmatrix} 0 \\ c_k \end{pmatrix} \quad \forall x \in \mathbf{R}^2,\ k = 1, 2,$$

with $B_1 \neq B_2$, then $\mathcal{G}_1$ would contain all the translations in the direction of $B_k\binom{0}{1}$, $k = 1, 2$ (see above). This is impossible, since we could obtain g_{W_2} by adjoining g_{W_1} with a suitable translation, which is an element of $\mathcal{G}_1$ ($\mathcal{G}_1$ would contain the whole translation group).

It remains to consider the following case. There is a boost B (fixed) such that for any $W \in \mathcal{W}_1$ either

$$g_W x = B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + B\begin{pmatrix} 0 \\ c \end{pmatrix} \quad \forall x \in \mathbf{R}^2,$$

$c \in \mathbf{R}$, or

$$g_W x = -x + a \quad \forall x \in \mathbf{R}^2,$$

$a \in \mathbf{R}^2$. We can assume that all translations by multiples of a spacelike vector v, which is inverted by $B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, are contained in $\mathcal{G}_1$ (see above). Let W be an arbitrary wedge and w a spacelike vector linearly independent of v such that $W+w \subset W$. Lemma 3.2.1.2 entails $g_{W+t_nw} \to \lim_n g_{W+t_nv} = g_W$ for any sequence $\{t_n \in \mathbf{R}\}$ decreasing or increasing to 0. This entails the continuity of $t \mapsto g_{W+tw}$ at $t = 0$ and, similarly, at any $t \in \mathbf{R}$. Hence, $g_{W+tw+sv} \equiv g_{W+tw} \equiv g_W$, modulo translations, i.e., the homogeneous part of $g_{W+tw+sv}$ is constant.

Theorem 3.2.1.6: *$W \mapsto g_W$ is continuous and the homogeneous part of g_W is independent of $W \in \mathcal{W}_1$. The translations in $\mathcal{G}_1$ form either a one-dimensional group of spacelike translations or the whole translation group. Furthermore, $\mathcal{G}_1 \cong \mathrm{IO}(1)$ or $\mathcal{G}_1$ is the semidirect product of the two-dimensional translation group and a group of order two, which is generated by an inversion.*

Proof: Let g_0 be the homogeneous part of the involutions g_W. First we consider the case $g_0x = -x \; \forall x \in \mathbf{R}^2$. We can assume that $\mathcal{G}_1$ contains spacelike translations by vectors $\binom{t_{n0}}{t_{n1}}$, which converge to 0 for $n \to \infty$ (Lemma 3.2.1.4). By considering a suitable subsequence, we can assume that $\{t_{n0}/t_{n1}\}$ converges ($|t_{n0}/t_{n1}| \leq 1$). It is now easy to see that $\mathcal{G}_1$ contains either a dense subset of the whole translation group or a dense subset of the translations in the direction of $\binom{\lim_n t_{n0}/t_{n1}}{1}$. Note that we can assume in the latter case that these translations are spacelike. We consider the first case. Let $W \in \mathcal{W}_1$ be an arbitrary wedge coherent to W_R. We can find increasing sequences of wedges $\{W_n\}$ and $\{V_n\}$ such that the wedges W_n, resp. V_n, can be obtained from W_R, resp. W, by translations in $\mathcal{G}_1$ and such that $W = \cup_n \overline{W_n} = \cup_n \overline{V_n}$. Lemma 3.2.1.2 shows $g_{W_n} \to \lim_n g_{V_n} = g_W$. Hence, in this case, any g_W, where $W \in \mathcal{W}_1$ is coherent to W_R, can be obtained from g_{W_R} by adjoining with the translation which maps W_R onto W. Similarly, any g_W, where $W \in \mathcal{W}_1$ is coherent to $g_{W_R}W_R$, can be obtained from g_{W_R} by adjoining with the translation which maps $g_{W_R}W_R$ onto W. Thus, $W \mapsto g_W$ is continuous and $\mathcal{G}_1$ is the semidirect product of the two-dimensional translation group and a group of order two, which is generated by an inversion.

Assume now that $\mathcal{G}_1$ contains a dense subset of translations in the direction of $v = \begin{pmatrix} \lim_n t_{n0}/t_{n1} \\ 1 \end{pmatrix}$. $t \mapsto g_{W_R}g_{W_R+tv}$ is continuous ($t \mapsto g_{W_R+tv}$ is continuous, according to the proof of Lemma 3.2.1.5) and $g_{W_R}g_{W_R+tv}$ are, for t in a dense subset of $\mathbf{R}$, translations in the direction of v (g_{W_R} acts as an inversion). Therefore, we obtain all translations in this direction.

In any case (also if the involutions g_W do not act as inversions; see the proof of Lemma 3.2.1.5), we can conclude that there is a spacelike vector v such that all translations in the direction of v are elements of $\mathcal{G}_1$. Again, we can show that $t \mapsto g_{W+tw}$ is continous for any spacelike w and any wedge W. Let $T_s \in \mathcal{G}_1$ be the translation by sv. $g_{W+tw+sv} = T_s g_{W+tw} T_s^{-1}$ shows that $W \mapsto g_W$ is continuous.

If the homogeneous part of g_W is of the form $B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then we obtain the translations by vectors which are inverted by $B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We cannot obtain any other translations, since $g_{W_1}g_{W_2}$ is a translation of this form for any $W_1, W_2 \in \mathcal{W}_1$ (Lemma 3.2.1.1). This shows $\mathcal{G}_1 \cong \mathrm{IO}(1)$. If g_W acts as an inversion, then at least a one-dimensional translation group is contained in $\mathcal{G}_1$. The continuity of $W \mapsto g_W$, resp. of $c \mapsto g_{W_R}g_{W_R+c}$, implies that we obtain either a one-dimensional translation group or the whole translation group.

Corollary 3.2.1.7: *If wedge duality is satisfied and if we choose the coordinate system suitably, then the modular conjugations act either as the inversions about the* *vertices of the corresponding wedges or they act as follows*

$$g_{W_R+\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)}x \;=\; \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} x + \begin{pmatrix}0\\ 2b\end{pmatrix} \quad \forall a,b\in\mathbf{R}, \forall x \in \mathbf{R}^2.$$

Proof: By choosing the coordinate system suitably, we can assume that the modular conjugations act by

$$g_{W_R+\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)}x \;=\; \begin{pmatrix}\pm 1 & 0\\ 0 & -1\end{pmatrix} x + c,$$

with $c \in \mathbf{R}^2$. Wedge duality, resp.

$$\begin{pmatrix}\pm a\\ -b\end{pmatrix} + c \;=\; g_{W_R+\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)}\begin{pmatrix}a\\ b\end{pmatrix} \;=\; \begin{pmatrix}a\\ b\end{pmatrix}$$

entails the claim.

3.2.2 Four-Dimensional Minkowski Space

Now we are ready to consider the $3+1$-dimensional Minkowski space.

Lemma 3.2.2.1: *Every involution $g \in \mathcal{P}$ acts, for a suitable choice of the coordinate system, as a diagonal matrix.*

Proof: Let g_0 be the homogeneous part of g. It suffices to prove the claim for g_0, since this would entail that g_0 is conjugate to g. Let l be a future-directed lightlike vector such that $g_0 l \neq \pm l$ (we can assume $g_0 \neq \pm 1$). We can either conclude

$$g_0 W[l, \pm g_0 l, 0] \;=\; W[l, \pm g_0 l, 0] \quad \text{or} \quad g_0 W[l, \pm g_0 l, 0] \;=\; W[l, \pm g_0 l, 0]'$$

($\pm g_0 l$ future-directed). By choosing the coordinate system suitably, we can assume $W_R = W[l, \pm g_0 l, 0]$ and, hence,

$$g_0 W_R \;=\; W_R \quad \text{or} \quad g_0 W_R \;=\; W_R',$$

resp. $g_0 = \pm BR\mathrm{diag}(1, \pm 1, \pm 1, \pm 1)R^{-1}$, where B is a boost in the direction of the x_1-axis and R is a rotation about the x_1-axis ($\pm g_0$ is an element of the invariance group of W_R).

Note $B = 1$ for $g_0 = \pm BR\mathrm{diag}(1, 1, \pm 1, \pm 1)R^{-1}$. The claim follows if we transform the coordinate system by a boost B' ($B'^2 = B^{-1}$) and by the rotation R.

Lemma 3.2.2.2: *$\mathcal{G}$ contains a spacelike translation.*

Proof: We will denote by (α, c) ($\alpha \in [0, 2\pi)$, $c \in \mathbf{R}^4$) elements of $\mathcal{G}$ of the form

$$(\alpha, c)x = R_1(\alpha)x + c \quad \forall x \in \mathbf{R}^4.$$

$g_{W_R+(0,1,0,0)^T} g_{W_R}$ maps W_R onto a wedge contained in W_R. Hence, it is, up to an element of the invariance group of W_R, a translation. We can show

$$(g_{W_R+(0,1,0,0)^T} g_{W_R})^2 = B\,(\alpha, c)$$

for a suitable $\alpha \in [0, 2\pi)$, a boost B, and a spacelike $c \in \mathbf{R}^4$ satisfying

$$\overline{W_R + c} \subset W_R.$$

We assume that $\mathcal{G}$ does not contain any spacelike translations and derive a contradiction. First we want to show that $\mathcal{G}$ contains an element (α_0, c_0) such that $\overline{W_R + c_0} \subset W_R$. Hence, we can assume $B'(\alpha', c') \in \mathcal{G}$ for a suitable boost $B' \neq 1$, $\alpha' \in [0, 2\pi)$, and $c' \in \mathbf{R}^4$ satisfying $\overline{W_R + c'} \subset W_R$. Translations in the direction of the x_0-axis and x_1-axis commute with transformations of the form (α, c). Therefore, we can transform the coordinate system with such translations. By choosing the coordinate system suitably, we can assume that the x_0- and x_1-components of c' are trivial. (Lemma 3.2.1.3). We have shown above that $\mathcal{G}$ contains elements $B''(\alpha'', c'')$ satisfying $\overline{W_R + c''} \subset W_R$. Note that $[B', [B', B''(0, c'')]] = [B', [B', (0, c'')]]$ is a translation by a spacelike vector with trivial x_2- and x_3-components (Lemma 3.2.1.3). Therefore, $[B'(\alpha',c'),[B'(\alpha',c'),B''(\alpha'',c'')]\in\mathcal{G}$ is a spacelike translation, since it is equal to $[B',[B',B''(0,c'')]]$, up to translations in the direction of the x_2- and x_3-axis. We obtain a contradiction for $B'\neq 1$ (we assumed that $\mathcal{G}$ does not contain any spacelike translations), i.e., $\mathcal{G}$ contains an element (α_0,c_0) such that $\overline{W_R+c_0}\subset W_R$.

By considering the transformations $(\alpha_0,c_0)^n$, $n\in\mathbf{N}$, we can conclude that $A_R=\{\alpha|(\alpha,c)\in\mathcal{G},\overline{W_R+c}\subset W_R\}$ is dense in $[0,2\pi)$. For any other wedge there is a similar set of elements of $\mathcal{G}$. Hence, we can choose the coordinate system suitably in the following.

In order to show $g=\mathrm{diag}(1,-1,\pm 1,\pm 1)\in\mathcal{G}$, for a suitable choice of the coordinate system, it suffices to prove that $\mathcal{G}$ contains a nontrivial involution in $\mathcal{P}^\uparrow$. Therefore, we only have to consider the case $g_W\in\mathcal{P}^\downarrow\ \forall W\in\mathcal{W}$. g_{W_R} is not an inversion, since, in general, $(g_{W_R}g_{W_R+c})^2$ is not a translation (we assumed that $\mathcal{G}$ does not contain any spacelike translations), i.e., there is a wedge $W\in\mathcal{W}$ such that $g_{W_R}W=W$ (according to Lemma 3.2.2.1, g_{W_R} acts as a diagonal matrix which has the eigenvalue 1). $g_Wg_{W_R}\in\mathcal{P}^\uparrow$ is an involution (g_{W_R} and g_W commute) and $g_Wg_{W_R}\neq 1$ ($g_Wg_{W_R}W=g_WW\neq W$, since wedge algebras are not maximally abelian).

Now we want to show that, up to a translation, $\mathcal{G}$ contains a nontrivial boost. By considering the wedge $B_1W_2^{(0)}$, where B_1 is a nontrivial boost in the direction of the x_1-axis, we can conclude that $B_1R_2(\alpha)B_1^{-1}$ are, up to translations and for angles α in a dense subset $A\subset[0,2\pi)$, elements of $\mathcal{G}$ (compare A_R). There is an angle $\alpha\in A$ such that $B_1R_2(\alpha)B_1^{-1}=RB$ for a rotation R and a boost $B\neq 1$. Note

$$\begin{aligned}&(gB_1R_2(\alpha)B_1^{-1}g^{-1})^{\mp 1}\\&\quad=(B_1^{-1}R_2(\pm\alpha)B_1)^{\mp 1}=B_1^{-1}R_2(-\alpha)B_1=(B_1R_2(\alpha)B_1^{-1})^{\mathrm{T}}=B^{\mathrm{T}}R^{\mathrm{T}},\end{aligned}$$

which implies

$$(gB_1R_2(\alpha)B_1^{-1}g^{-1})^{\mp 1}(B_1R_2(\alpha)B_1^{-1}) = B^2 \neq 1.$$

This is, up to a translation, an element $h \in \mathcal{G}$. By choosing the coordinate system suitably, we can assume that B^2 is a boost in the direction of the x_1-axis. Choose $(\alpha, c) \in \mathcal{G}$ such that $\overline{W_R + c} \subset W_R$. Then

$$[B^2, [B^2, (0, c)]] \equiv [h, [h, (\alpha, c)]] \in \mathcal{G},$$

modulo translations in the direction of the x_2-axis and x_3-axis. However, $[B^2, [B^2, (0, c)]]$ is a spacelike translation with trivial x_2- and x_3-component (Lemma 3.2.1.3). This implies that $[h, [h, (\alpha, c)]] \in \mathcal{G}$ is a spacelike translation.

Lemma 3.2.2.3: *$\mathcal{G}$ contains two spacelike translations in directions which are orthogonal.*

Proof: The proof of Lemma 3.2.2.2 implies that $\mathcal{G}$ contains an element (α, c) which is a product of g_W's where W is coherent to W_R and such that $\overline{W_R + c} \subset W_R$. We can assume that $\mathcal{G}$ contains a translation $T \neq 1$ in the direction of the x_3-axis (Lemma 3.2.2.2). If $W \in \mathcal{W}$ is coherent to W_R, then g_W commutes with T, i.e., $g_W x = \mathrm{diag}(A, 1)x + a$ for a 3×3-matrix A and $a \in \mathbf{R}^4$ with trivial x_3-component ($g_W^2 = 1$). We can conclude $\alpha = 0$ and that the x_3-component of c is trivial. c is evidently spacelike.

Lemma 3.2.2.4: *$\mathcal{G}$ contains a 3-dimensional group of spacelike translations.*

Proof: We can assume that $\mathcal{G}$ contains nontrivial translations in the direction of the x_2-axis and the x_3-axis. g_W, where $W \in \mathcal{W}$ is a wedge coherent to W_R or W_R', commutes with these translations implying $g_W \equiv \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, modulo translations in the direction of the x_0-axis and the x_1-axis (recall $g_W^2 = 1$). A is a 2×2-matrix. We can apply Theorem 3.2.1.6 (g_W leaves the set of wedges which are coherent to W_R or W_R' invariant) to obtain a one-dimensional group of spacelike translations.

By using similar arguments, we will finally get a 3-dimensional group of spacelike translations contained in $\mathcal{G}$.

Lemma 3.2.2.5: *The subgroup of $\mathcal{G}$ consisting of the translations is either 3-dimensional or the whole translation group.*

Proof: Assume that $\mathcal{G}$ does not contain the whole translation group. For a suitable choice of the coordinate system, the translations by vectors of the form $(0, x_1, x_2, x_3)^{\mathrm{T}}$ are contained in $\mathcal{G}$, according to Lemma 3.2.2.4. We will call this group $\mathcal{T}$. The adjoint action of g_W, $W \in \mathcal{W}$, must leave $\mathcal{T}$ invariant, since we assumed that $\mathcal{G}$ does not contain all the translations. We want to show that $\mathcal{G}$ does not contain any other translations. Hence, we assume that there is a translation $T \in \mathcal{G} \backslash \mathcal{T}$ and we will derive a contradiction.

If $W = RW_R + a$, for a rotation R and $a \in \mathbf{R}^4$, then $g_W W$ is coherent to W', since g_W commutes with the two-dimensional group of translations which leaves W invariant (compare the proof of Lemma 3.2.2.4). Therefore, we can apply Theorem 3.2.1.6, i.e., $c \mapsto g_{W+c}$ is continuous. Since the adjoint action of g_W leaves $\mathcal{T}$ invariant, we conclude (see also Lemma 3.2.1.1)

$$g_W \equiv R \operatorname{diag}(\pm 1, -1, 1, 1) R^{-1}, \qquad (3.2.2.1)$$

modulo translations (we obtain the same sign for fixed R and arbitrary $a \in \mathbf{R}^4$, according to Theorem 3.2.1.6). However, $g_W \in \mathcal{P}^{\downarrow}$ (i.e., $-$ in (3.2.2.1)) would entail that the translation $(g_W T g_W)T^{-1} = g_W(T g_W T^{-1}) = g_W g_{W+c_0}$ ($c_0 \in \mathbf{R}^4$ suitably) is not an element of $\mathcal{T}$ and, because of the continuity of $c \mapsto g_W g_{W+c}$, $\mathcal{G}$ would contain the whole translation group ($g_W g_{W+c}$ are translations). We have shown

$$g_{RW_R+a} = g_W \equiv R\,\mathrm{diag}(1,-1,1,1)R^{-1}.$$

If $W \in \mathcal{W}$ cannot be obtained from W_R by rotations and translations, then we can also show that $c \mapsto g_{W+c}$ is continuous and that $g_W g_{W+c}$ are translations ($W + c$ can be obtained from W by translations in $\mathcal{T}$, since the subgroup of $\mathcal{T}$ leaving W invariant cannot be two-dimensional, i.e., it is only one-dimensional). As above follows

$$g_W x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & a_6 \\ 0 & a_7 & a_8 & a_9 \end{pmatrix} x + d, \quad \forall x \in \mathbf{R}^4 \tag{3.2.2.2}$$

(recall $g_W \mathcal{T} g_W = \mathcal{T}$). $g_W^2 = 1$ implies that d has a trivial x_0-component. Since any g_W is of the form (3.2.2.2) (also if $W = RW_R + a$ for a rotation R and $a \in \mathbf{R}^4$; see above), this yields a contradiction (every g_W, i.e., also $T \in \mathcal{G}$, acts trivially on the x_0-component).

Lemma 3.2.2.6: *Let $\mathcal{R}$ be a Lie group. If $U_1(t), \ldots, U_n(t)$, $t \in \mathbf{R}$, are continuous one-parameter subgroups, then they generate a subgroup $\mathcal{R}_0$ of $\mathcal{R}$, which is a Lie group itself.*

Proof: Let $V_1(t) = \exp(tA_1), \ldots, V_m(t) = \exp(tA_m)$ be a set of one-parameter subgroups of $\mathcal{R}_0$ such that $\{A_1, \ldots, A_m\}$ span a maximal subspace V of the Lie algebra of $\mathcal{R}$. We have to show that V contains $[A_i, A_j]$ for $i, j = 1, \ldots, m$. Note

$$(\exp(tA_i)A_j \exp(-tA_i) - A_j)/t \in V,$$

i.e. V contains a sequence converging to $[A_i, A_j]$ (let $t \to 0$). However, subspaces of finite-dimensional vector spaces are closed implying $[A_i, A_j] \in V$. Hence, the elements of the form

$$V_1(t_1) \cdot \ldots \cdot V_m(t_m), \quad t_1, \ldots, t_m \in \mathbf{R},$$

generate the connected Lie group whose Lie algebra is V.

In the proof of the following lemma we will use the Lie algebra of the Lorentz group. Let K_1, K_2, K_3 be generators of the boosts and S_1, S_2, S_3 be generators of the rotations. We can assume the following commutation relations:

$$[S_i, S_j] = \varepsilon_{ijk} S_k, \; [S_i, K_j] = \varepsilon_{ijk} K_k, \; [K_i, K_j] = -\varepsilon_{ijk} S_k, \tag{3.2.2.3}$$

where

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$K_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; K_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Before we can determine the possible groups $\mathcal{G}$ which can occur, we have to consider the following case.

Lemma 3.2.2.7: *Assume that $\mathcal{G}$ contains the whole translation group and that there is a wedge $W_0 \in \mathcal{W}$ such that $g_{W_0} \in \mathcal{P}^{\downarrow}$. Then, either $\mathcal{G}$ contains the rotation group (transform the coordinate system suitably if necessary) or $\mathcal{G}$ consists of the transformations whose homogeneous part leaves $\{\pm l\}$, for a lightlike vector $l \neq 0$, invariant.*

Proof: We can assume $W_0 = W_R$ and, modulo translations,

$$g_{W_R} \equiv \operatorname{diag}(-1,-1,1,1)$$

(g_{W_R} commutes with the translations which leave W_R invariant; use Lemma 3.2.1.1). It is easy to show

$$g_{R_1(\alpha)W_2^{(0)}} \equiv R_1(\alpha)\operatorname{diag}(\pm 1, 1, -1, 1)R_1(-\alpha)$$

($g_{R_1(\alpha)W_2^{(0)}}$ commutes with $\operatorname{diag}(-1,-1,1,1) \in \mathcal{G}$ and the translations which leave $R_1(\alpha)W_2^{(0)}$ invariant). This implies that $\mathcal{G}$ contains the rotations about the x_1-axis ($(g_{R_1(\alpha)W_2^{(0)}} g_{R_1(\alpha')W_2^{(0)}})^2 = R_1(4\alpha - 4\alpha')$). Let L be the Lie algebra of the Lie group generated by the continuous one-parameter subgroups of $\mathcal{G}$ (see Lemma 3.2.2.6). We have shown $S_1 \in L$.

Now we want to prove that, if $\mathcal{G}$ does not contain the whole rotation group for a suitable choice of the coordinate system, there is an element $aS_3 + bK_2 \in L$, $a, b \in \mathbf{R}$, $b \neq 0$. Since $\mathcal{G}$ contains the whole translation group, we can conclude

$$g_{R_2(\varphi)W_R} \equiv R_2(\varphi)e^{\alpha K_1}\operatorname{diag}(\pm 1, -1, 1, 1)R_2(-\varphi) \in \mathcal{G},$$

for a suitable α depending on φ (see Lemma 3.2.1.1; $\alpha = 0$ for $-$). $\alpha = 0$ would imply

$$R_2(4\varphi) = (R_2(\varphi)\operatorname{diag}(\pm 1, -1, 1, 1)R_2(-\varphi)\operatorname{diag}(-1,-1,1,1))^2 \in \mathcal{G},$$

i.e., for a suitable choice of $\varphi \neq n\pi/2$ ($n \in \mathbf{Z}$), we can assume $\alpha \neq 0$ ($\mathcal{G}$ does not contain the whole rotation group), hence

$$R_2(\varphi)e^{\alpha K_1}\mathrm{diag}(1,-1,1,1)R_2(-\varphi) \in \mathcal{G}.$$

Furthermore, $\varphi \neq n\pi/2$ entails $R_2(-\varphi)S_1R_2(\varphi) = cS_1 + c'S_3$ for $c, c' \in \mathbf{R}$, $c' \neq 0$ (see (3.2.2.3)).

$$\begin{aligned}
&(R_2(\varphi)e^{\alpha K_1}\mathrm{diag}(1,-1,1,1)R_2(-\varphi))S_1(R_2(\varphi)e^{\alpha K_1}\mathrm{diag}(1,-1,1,1)R_2(-\varphi)) \\
&\quad = \quad R_2(\varphi)e^{\alpha K_1}\mathrm{diag}(1,-1,1,1)(cS_1 + c'S_3)\mathrm{diag}(1,-1,1,1)e^{-\alpha K_1}R_2(-\varphi) \\
&\quad = \quad R_2(\varphi)e^{\alpha K_1}(cS_1 - c'S_3)e^{-\alpha K_1}R_2(-\varphi) \\
&\quad \overset{(3.2.2.3)}{=} \quad R_2(\varphi)cS_1R_2(-\varphi) - c'R_2(\varphi)e^{\alpha K_1}S_3e^{-\alpha K_1}R_2(-\varphi) \\
&\quad \overset{(3.2.2.3)}{=} \quad R_2(\varphi)cS_1R_2(-\varphi) - c'R_2(\varphi)(\cosh\alpha\, S_3 - \sinh\alpha\, K_2)R_2(-\varphi) \\
&\quad \overset{(3.2.2.3)}{=} \quad R_2(\varphi)cS_1R_2(-\varphi) - c'\cosh\alpha\, R_2(\varphi)S_3R_2(-\varphi) + c'\sinh\alpha\, K_2,
\end{aligned}$$

which is an element of L, is a linear combination of S_1, S_3 and K_2, according to (3.2.2.3). The claim follows from $c'\sinh\alpha \neq 0$ and $S_1 \in L$.

It is easy to derive from (3.2.2.3)

$$e^{\lambda K_1}S_3e^{-\lambda K_1} = \cosh\lambda\, S_3 - \sinh\lambda\, K_2$$

and

$$e^{\lambda K_1}K_2e^{-\lambda K_1} = \cosh\lambda\, K_2 - \sinh\lambda\, S_3.$$

For $|b| < |a|$, we choose λ such that $\sinh\lambda/\cosh\lambda = -b/a$, i.e. such that $e^{\lambda K_1}S_3e^{-\lambda K_1} = \cosh\lambda/a\,(aS_3 + bK_2) \in L$. Hence, $S_1, S_3 \in L$, resp. the rotation group is contained in $\mathcal{G}$, after transforming the coordinate system with the boost $e^{\lambda K_1}$ (this boost commutes with rotations about the x_1-axis). Similarly, for $|b| > |a|$, we can assume that the boosts in the direction of the x_2-axis and the rotations about the x_1-axis are contained in $\mathcal{G}$. Since $g_{W_2^{(0)}}$ must commute with these boosts, we can conclude $g_{W_2^{(0)}} = \mathrm{diag}(-1,1,-1,1)$ and, as above, it follows that the rotations about the x_2-axis are contained in $\mathcal{G}$. Again we have shown that $\mathcal{G}$ contains the whole rotation group (for a suitable choice of the coordinate system). Hence, we only have to consider the two cases $S_3 \pm K_2 \in L$ ($|a| = |b|$). We assume $S_3 + K_2 \in L$ (the other case follows similarly). Note

$$[S_1, S_3 + K_2] \;=\; S_2 - K_3 \;\in\; L.$$

It is well-known that the stabilizer group $S(l_{1+})$ of l_{1+} in $\mathcal{L}_+^\uparrow$ is isomorphic to IO(2) and its Lie algebra is generated by S_1, S_3+K_2, S_2-K_3. Since $\mathcal{G}$ contains $S(l_{1+})$, $\mathrm{diag}(-1,-1,1,1) \in \mathcal{L}_+^\downarrow$ and $\mathrm{diag}(1,1,-1,1) \in \mathcal{L}_-^\uparrow$ (recall $g_{W_2^{(0)}} \equiv \mathrm{diag}(\pm 1,1,-1,1)$ and note that $\mathcal{G}$ does not contain all the rotations about the x_2-axis excluding $-$), which leave $\{l_{1+}, -l_{1+}\}$ invariant, it contains all elements of the Lorentz group $\mathcal{L}$ which leave this set invariant. Hence, the claim is proven if the homogeneous part of g_W leaves $\{l_{1+}, -l_{1+}\}$ invariant, for all $W \in \mathcal{W}$.

We can assume that there is a wedge $W \in \mathcal{W}$ such that $h_W l_{1+} \notin \{l_{1+}, -l_{1+}\}$ for the homogeneous part h_W of g_W, i.e., $h_W l_{1+}$ is not a multiple of l_{1+} ($h_W^2 = 1$). The adjoint action of $h_W \in \mathcal{G}$ cannot leave $S(l_{1+})$ invariant (multiples of l_{1+} are the only fixed points of $S(l_{1+})$). L even contains an element $A = c_1K_1 + c_2K_2 + c_3K_3 + c_4S_1 + c_5S_2 + c_6S_3$, which is not a linear combination of $K_1, S_1, S_3+K_2, S_2-K_3$ (elements of the Lie group corresponding to the latter four elements map l_{1+} onto multiples of l_{1+}). By adjoining A with the reflections $\mathrm{diag}(-1,-1,1,1) \in \mathcal{G}$ and $\mathrm{diag}(1,1,-1,1) \in \mathcal{G}$ (recall $g_{W_2^{(0)}} \equiv \mathrm{diag}(1,1,-1,1)$), we can show $c_1K_1 - c_2K_2 - c_3K_3 + c_4S_1 - c_5S_2 - c_6S_3 \in L$ and $c_1K_1 - c_2K_2 + c_3K_3 - c_4S_1 + c_5S_2 - c_6S_3 \in L$, resp.

$$c_1K_1,\; c_2K_2 + c_6S_3,\; c_3K_3 + c_5S_2 \in L.$$

Not all of these three elements are linear combinations of $K_1, S_1, S_3 + K_2, S_2 - K_3$. It follows now that either $c_2K_2 + c_6S_3$ is not a multiple of $S_3 + K_2$ or $c_3K_3 + c_5S_2$ is not a multiple of $S_2 - K_3$, i.e., $K_2, S_3 \in L$ or $K_3, S_2 \in L$. This proves the claim.

Theorem 3.2.2.8: *$\mathcal{G} = \mathcal{P}_+$, $\mathcal{G} \cong \mathrm{IO}(3)$, $\mathcal{G} \cong \mathrm{IO}(3) \times \mathbf{Z}_2$, $\mathcal{G} \cong \mathrm{IO}(3) \times \mathrm{IO}(1)$, or $\mathcal{G}$ consists of the transformations whose homogeneous part leaves $\{\pm l\}$, for a lightlike vector $l \neq 0$, invariant.*

Proof: We assume first that $\mathcal{G}$ does not contain the whole translation group, i.e. the translations in $\mathcal{G}$ form a 3-dimensional subgroup $\mathcal{T}$, according to Lemma 3.2.2.5. For a suitable choice of the coordinate system, $\mathcal{T}$ consists of the translations by vectors of the form $(0, x_1, x_2, x_3)^{\mathrm{T}}$. As above, we can prove (3.2.2.1). Therefore, we obtain, modulo the time-inversion, all rotations. $\mathcal{G}$ even contains the whole rotation group. An element of the Lorentz group $\mathcal{L}_+^\uparrow$ leaves $\mathcal{T}$ invariant if and only if it is a rotation. Hence, $\mathcal{G}$ lies in the group generated by $\mathrm{diag}(-1,-1,1,1) \in \mathcal{L}_+^\downarrow$, $\mathrm{diag}(1,-1,1,1) \in \mathcal{L}_-^\uparrow$, the rotations, and $\mathcal{T}$, since the adjoint action of g_W, $W \in \mathcal{W}$, leaves $\mathcal{T}$ invariant. This group is isomorphic to $\mathrm{IO}(3) \times \mathbf{Z}_2$. Now it follows easily $\mathcal{G} \cong \mathrm{IO}(3)$, resp. $\mathcal{G} \cong \mathrm{IO}(3) \times \mathbf{Z}_2$, if $\mathcal{G}$ contains the time-inversion (note $g_{W_R} \equiv \mathrm{diag}(\pm 1, -1, 1, 1) \in \mathcal{G}$).

Assume now that $\mathcal{G}$ contains the whole translation group and $g_W \in \mathcal{P}^\uparrow \; \forall W \in \mathcal{W}$. We want to show $g = \mathrm{diag}(1,-1,-1,-1) \in \mathcal{G}$, for a suitable choice of the coordinate system. After applying a boost in the direction of the x_1-axis we can assume $g_{W_R} \equiv \mathrm{diag}(1,-1,1,1) \in \mathcal{G}$. After applying another boost, this time in the direction of the x_2-axis, we can assume $g_{W_2^{(0)}} \equiv \mathrm{diag}(1,1,-1,1) \in \mathcal{G}$. This boost commutes with $\mathrm{diag}(1,-1,1,1)$ implying $\mathrm{diag}(1,-1,1,1)$, $\mathrm{diag}(1,1,-1,1) \in \mathcal{G}$. Finally, we can assume, in addition, $\mathrm{diag}(1,1,1,-1) \in \mathcal{G}$, resp. $\mathrm{diag}(1,-1,-1,-1) \in \mathcal{G}$.

Note that $g_W W$ and W' are coherent $\forall W \in \mathcal{W}$ (g_W commutes with the translations which leave W invariant; see the proof of Lemma 3.2.2.4). Thus, to any rotation R, there is a translation $T \in \mathcal{G}$ such that

$$\mathcal{R}(TgRW_R) \;=\; \mathcal{R}(TRW_R') \;=\; \mathcal{R}(g_{RW_R}RW_R) \;=\; \mathcal{R}(RW_R)'.$$

This implies that g_{RW_R} commutes with $Tg \in \mathcal{G}$ (the modular conjugations of $\mathcal{R}(RW_R)$ and $\mathcal{R}(RW_R)'$ agree) and is, therefore, of the form

$$g_{RW_R} \;\equiv\; R\,\mathrm{diag}(\pm 1, -1, 1, 1)R^{-1}$$

(Lemma 3.2.1.1). Thus, we obtain again the whole rotation group.

According to Lemma 3.2.2.7 and the preceding paragraphs, it remains to consider the case in which $\mathcal{G}$ contains all translations and the rotation group.

If the adjoint action of g_W, $W \in \mathcal{W}$, does not leave the group generated by the rotations and the translations invariant, then there is a one-dimensional subgroup $R(\alpha)$ of the rotation group such that

$$g(\alpha) \;=\; g_W R(\alpha) g_W \in \mathcal{G} \cap \mathcal{P}_+^\uparrow$$

is not, modulo translations, a rotation for all α. There are suitable rotations R_α, R'_α, and $t_\alpha \in \mathbf{R}$ such that $g(\alpha) \equiv R_\alpha v_1(t_\alpha) R'_\alpha$ (we denote by $v_1(t)$ the boosts in the direction of the x_1-axis). $\alpha \mapsto |t_\alpha|$ is continuous. We can assume $t_\alpha \geq 0$ (if necessary, replace R_α by $R_\alpha R_3(\pi)$ and R'_α by $R_3(\pi)R'_\alpha$), i.e., $\alpha \mapsto t_\alpha$ is continuous and not identically 0. Hence, $\mathcal{G}$ contains the whole group of boosts in the direction of the x_1-axis. This implies $\mathcal{G} \supset \mathcal{P}_+^\uparrow$. Now it is not difficult to conclude $\mathcal{G} = \mathcal{P}_+$ (see [18] or the proof of Theorem 3.3.1).

We will complete the proof of the claim by showing $\mathcal{G} \cong \mathrm{IO}(3) \times \mathrm{IO}(1)$ in the case that, for any $W \in \mathcal{W}$, the adjoint action of g_W leaves the group generated by the rotations and the translations invariant. We still assume that $\mathcal{G}$ contains all translations and rotations. $\mathcal{G}$ only contains transformations which lie in the group generated by $\mathrm{diag}(-1,-1,1,1) \in \mathcal{L}_+^{\downarrow}$, $\mathrm{diag}(1,-1,1,1) \in \mathcal{L}_-^{\uparrow}$, the rotations, and the translations, since the adjoint action of the homogeneous part of g_W leaves the rotation group invariant (compare the first paragraph of the proof). Therefore, we only have to prove $\mathrm{diag}(1,-1,1,1) \in \mathcal{G}$ and that there is a wedge $W \in \mathcal{W}$ such that $g_W \in \mathcal{P}^{\downarrow}$.

For wedges which cannot be obtained from W_R by rotations and translations we can derive (3.2.2.2), since the adjoint action of the homogeneous part of g_W leaves the rotation group invariant and these wedges are invariant with respect to certain translations by vectors with nontrivial x_0-component (these translations lie in $\mathcal{G}$). $g_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},0]}$, $a,b \neq 0$, $a^2+b^2=1$, commutes with the translations in the direction of the x_3-axis implying, together with (3.2.2.2),

$$g_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},0]} \equiv R'\mathrm{diag}(1,\pm 1,1,1)$$

for a rotation R' about the x_3-axis (elements of the Lorentz group which act trivially on the x_0- and x_3-component are, possibly up to a reflection, a rotation about the x_3-axis). However,

$$g_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},0]} W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},0]$$

is coherent to $W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},0]'$ ($\mathcal{G}$ contains the whole translation group), which leaves, after a short computation, only the possibility

$$g_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},0]} \equiv \mathrm{diag}(1,-1,1,1) \in \mathcal{G}. \tag{3.2.2.4}$$

Since not all of the g_W can act as in (3.2.2.2) ($\mathcal{G}$ contains timelike translations; see the conclusion after (3.2.2.2)), there is a wedge $W \in \mathcal{W}$ such that $g_W \in \mathcal{P}^{\downarrow}$. This proves $\mathcal{G} \cong \mathrm{IO}(3) \times \mathrm{IO}(1)$.

We will use that any wedge is, up to translations and rotations, of the form $W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},0]$ ($a^2+b^2=1$, $a \neq 0$). This can be proven as follows. Note that two arbitrary vectors can always be rotated such that they have trivial x_3-components. Let $(1,a_1,b_1,0)^{\mathrm{T}}$, $(1,a_2,b_2,0)^{\mathrm{T}}$ be lightlike vectors with $b_1 < b_2$. After a rotation by π about the x_3-axis we obtain the vectors $(1,-a_1,-b_1,0)^{\mathrm{T}}$, $(1,-a_2,-b_2,0)^{\mathrm{T}}$. The Intermediate Value Theorem entails that we obtain vectors of the form $(1,c_1,b,0)^{\mathrm{T}},(1,c_2,b,0)^{\mathrm{T}}$, after a rotation $R_3(\alpha)$ for a suitable $0<\alpha<\pi$ ($b_1 < b_2$ but $-b_1 > -b_2$). However, the vectors are different and lightlike, i.e., $c_1 = -c_2$.

We postpone the case $\mathcal{G} = \mathcal{P}_+$ (see 3.3).

Corollary 3.2.2.9: *Assume wedge duality. If $\mathcal{G}$ consists of the transformations whose homogeneous part leaves $\{\pm l\}$, for a lightlike vector $l \neq 0$, invariant, then g_W, $W \in \mathcal{W}$, is the uniquely determined involution which maps W onto W' and whose homogeneous part leaves $\{\pm l\}$ invariant.*

In the other cases, if $\mathcal{G} \neq \mathcal{P}_+$ and the coordinate system is chosen such that $\mathcal{G}$ contains the rotations, the geometric action g_W of the modular conjugations is given by (up to a uniquely determined translation)

$$g_{RW_R+c} \equiv R\,\mathrm{diag}(\pm 1,-1,1,1)R^{-1}, \qquad (3.2.2.5)$$

resp. by

$$g_{RW[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},c]} \equiv R\,\mathrm{diag}(1,-1,1,1)R^{-1} \qquad (3.2.2.6)$$

or

$$g_{RW[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},c]} \equiv R\,\mathrm{diag}(-1,-1,-1,1)R^{-1}, \qquad (3.2.2.7)$$

for arbitrary $c \in \mathbf{R}^4$*, rotations* R*, and* $a, b \in \mathbf{R}$ *satisfying* $a^2 + b^2 = 1$ *and* $a, b \neq 0$*.*

$\mathcal{G} \cong \mathrm{IO}(3) \times \mathrm{IO}(1)$*, resp.* $\mathcal{G} \cong \mathrm{IO}(3) \times \mathbf{Z}_2$ *and* $\mathcal{G} \cong \mathrm{IO}(3)$*, imply* $-$*, resp.* $+$*, in (3.2.2.5).* $\mathcal{G} \cong \mathrm{IO}(3) \times \mathrm{IO}(1)$ *and* $\mathcal{G} \cong \mathrm{IO}(3)$ *exclude (3.2.2.7).*

Proof: Let $\mathcal{G}$ consist of the transformations whose homogeneous part leaves $\{\pm l\}$ invariant. It is sufficient to show for $W = W_R$, $l = (1, a_1, a_2, a_3)^{\mathrm{T}}$, $a_1^2 + a_2^2 + a_3^2 = 1$, that g_W is uniquely determined. $\mathcal{G}$ contains the whole translation group implying $g_{W_R} \equiv B\,\mathrm{diag}(\pm 1, -1, 1, 1)$ for a boost in the direction of the x_1-axis ($B = 1$ for $-$). Wedge duality implies $g_{W_R} = B\,\mathrm{diag}(\pm 1, -1, 1, 1)$. If $l = l_{1\pm}$, then there is only the possibility $g_{W_R} = \mathrm{diag}(-1, -1, 1, 1)$. If $l \neq l_{1\pm}$, we can conclude $g_{W_R} = B\,\mathrm{diag}(1, -1, 1, 1)$ and

$$B\,\mathrm{diag}(1, -1)(1, a_1)^{\mathrm{T}} \;=\; (1, a_1)^{\mathrm{T}}.$$

There is only one boost B such that the timelike vector $(1, a_1)^{\mathrm{T}}$ is an eigenvector of $B\,\mathrm{diag}(1, -1)$ to the eigenvalue 1, i.e., g_W, $W \in \mathcal{W}$, is uniquely determined.

It remains to consider the cases $\mathcal{G} \cong \mathrm{IO}(3) \times \mathrm{IO}(1)$, $\mathcal{G} \cong \mathrm{IO}(3)$, and $\mathcal{G} \cong \mathrm{IO}(3) \times \mathbf{Z}_2$ (Theorem 3.2.2.8). In the first case, it was already shown

$$g_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},c]} \;\equiv\; \mathrm{diag}(1, -1, 1, 1)$$

for $c \in \mathbf{R}^4$ and $b \neq 0$ (see (3.2.2.4)). The adjoint action of the homogeneous part of g_{W_R+c}, $c \in \mathbf{R}^4$, leaves the rotation group invariant and $\mathcal{G}$ contains the whole translation group implying

$$g_{W_R+c} \;\equiv\; \mathrm{diag}(\pm 1, -1, 1, 1)$$

($+\ \forall c \in \mathbf{R}^4$ or $-\ \forall c \in \mathbf{R}^4$, according to Theorem 3.2.1.6). Since $\mathcal{G}$ contains timelike translations, we can even conclude

$$g_{W_R+c} \;\equiv\; \mathrm{diag}(-1, -1, 1, 1) \in \mathcal{G}. \tag{3.2.2.8}$$

Otherwise, every g_W, $W \in \mathcal{W}$, would be of the form (3.2.2.2), since $\mathcal{G}$ contains all rotations and all translations and since every g_W can be obtained from (3.2.2.4) or (3.2.2.8) by such transformations (see the conclusion after (3.2.2.2)). The last remark also implies that the homogeneous part of any g_W, $W \in \mathcal{W}$, is uniquely determined in the case $\mathcal{G} \cong \mathrm{IO}(3) \times \mathrm{IO}(1)$. Wedge duality and $g_W^2 = 1$ show again that there is only one possible action for g_W.

In the latter two cases $\mathcal{G} \cong \mathrm{IO}(3)$ and $\mathcal{G} \cong \mathrm{IO}(3) \times \mathbf{Z}_2$, we can conclude, using familiar arguments,

$$g_{RW_R+c} \equiv R\,\mathrm{diag}(1,-1,1,1)R^{-1}, \tag{3.2.2.9}$$

for all $c \in \mathbf{R}^4$ and rotations R, since $\mathcal{G}$ does not contain any timelike translations. (3.2.2.6) and (3.2.2.7) can be derived similar as (3.2.2.4).

Finally, we want to show that $\mathcal{G} \cong \mathrm{IO}(3)$ excludes (3.2.2.7). However, in this case, $\mathcal{G}$ has only two conneted components, which would contradict

$$\mathrm{diag}(-1,-1,-1,1) \in \mathcal{G} \cap \mathcal{L}_-^{\downarrow} \quad \text{and} \quad g_{W_R} \equiv \mathrm{diag}(1,-1,1,1) \in \mathcal{G} \cap \mathcal{L}_-^{\uparrow}.$$

Corollary 3.2.2.10: *If wedge duality and the continuity of $W \mapsto J_W$ are satisfied (in the sense of [18]) and if, furthermore, $\mathcal{G} \neq \mathcal{P}_+$, then $\mathcal{G} \cong \mathrm{IO}(3)$ and, for a suitable choice of the coordinate system (such that $\mathcal{G}$ contains the rotations),*

$$g_{W[(1,a,b,0)^T,(1,-a,b,0)^T,0]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for arbitrary $a, b \in \mathbf{R}$ satisfying $a^2 + b^2 = 1$ and $a \neq 0$. In general, the modular conjugations act by

$$g_{RTW[(1,a,b,0)^T,(1,-a,b,0)^T,0]} = RT g_{W[(1,a,b,0)^T,(1,-a,b,0)^T,0]} T^{-1} R^{-1},$$

where R are arbitrary rotations and T are arbitrary translations.

Proof: First we assume that $\mathcal{G}$ contains the rotation group. Assume that we can choose $a_n, b_n \in \mathbf{R}$ such that $a_n \to a \neq 0$, $b_n \to b$, $a_n^2 + b_n^2 = 1$, $g_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},c]}$ acts as in (3.2.2.5) or (3.2.2.6), and the involutions $g_{W[(1,a_n,b_n,0)^{\mathrm{T}},(1,-a_n,b_n,0)^{\mathrm{T}},c]}$ act as in (3.2.2.7) ($c \in \mathbf{R}^4$ arbitrary but fixed). We will find a contradiction. Let $T \in \mathcal{J}$ act as a translation by $(0,0,1,0)^{\mathrm{T}}$.

$$J_{W[(1,a_n,b_n,0)^{\mathrm{T}},(1,-a_n,b_n,0)^{\mathrm{T}},c]}T\,J_{W[(1,a_n,b_n,0)^{\mathrm{T}},(1,-a_n,b_n,0)^{\mathrm{T}},c]}T$$

are internal symmetries strongly converging to

$$J_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},c]}T\,J_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},c]}T,$$

which acts as a translation by $(0,0,2,0)^{\mathrm{T}}$. However, this provides a contradiction (compare the proof of Lemma 3.2.1.2). We have shown that the set

$$\{b \in (-1,1) \,|\, g_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},c]} \text{ acts as in (3.2.2.7)}\}$$

is closed in $(-1,1)$. Similar arguments prove that the set

$$\{b \in (-1,1) \,|\, g_{W[(1,a,b,0)^{\mathrm{T}},(1,-a,b,0)^{\mathrm{T}},c]} \text{ acts as in (3.2.2.5) or (3.2.2.6) }\}$$

is closed and non-empty ($b = 0$ is an element). The connectedness of $(-1,1)$ excludes (3.2.2.7). Similar arguments show that, if $\mathcal{G}$ contains the whole translation group, then we have to choose $+$ in (3.2.2.5). Therefore, the case $\mathcal{G} \cong \mathrm{IO}(3) \times \mathrm{IO}(1)$ cannot occur (Corollary 3.2.2.9). It follows $\mathcal{G} \cong \mathrm{IO}(3)$, since we have proven $g_W \in \mathcal{P}_-^{\uparrow}$ for all $W \in \mathcal{W}$ (wedge duality implies $+$ in (3.2.2.5) if $\mathcal{G}$ does not contain the whole translation group).

If $\mathcal{G}$ consists of the transformations whose homogeneous part leaves $\{\pm l\}$, for a lightlike vector $l \neq 0$, invariant, then we can proceed similar as above to exclude this case (note that $\mathcal{G}$ contains the whole translation group and that there are wedges $W \in \mathcal{W}$ such that $g_W \in \mathcal{P}^\uparrow$ and wedges such that $g_W \in \mathcal{P}^\downarrow$).

3.3 Transitive Action

We will show in this section that the strong CGMA and the transitivity of the action of the modular conjugations imply that the modular conjugations act as in [4, 5], i.e., in particular,

$$J_{W_R}\mathcal{R}(W)J_{W_R} \;=\; \mathcal{R}(g_{W_R}W) \quad \forall W \in \mathcal{W} \tag{3.3.1}$$

with

$$g_{W_R}x \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x \quad \forall x \in \mathbf{R}^4. \tag{3.3.2}$$

The following theorem was already proven in [18, Proposition 4.2.10]. We give a shorter proof which is, however, only applicable to this particular situation. Recall that the invariance group of W_R can be generated by the boosts in the direction of the x_1-axis, by the rotations about the x_1-axis, and by elements of the form $\mathrm{diag}(\pm 1, \pm 1, \pm 1, 1)$.

Theorem 3.3.1: *Assume the strong CGMA and that the modular conjugations act transitively on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. Then $\mathcal{G} = \mathcal{P}_+$, wedge duality is fulfilled, and the modular conjugations act as in (3.3.1).*

Proof: It suffices to prove $\mathcal{G} \supset \mathcal{P}_+^\uparrow$. (3.3.2) follows then easily from the uniqueness of the modular objects, i.e., from the fact that g_{W_R} commutes with elements of $\mathcal{P}_+^\uparrow$ which leave W_R invariant (see the proof of [18, Proposition 4.2.10]). Note that any involution acts, for a suitable choice of the coordinate system, as a diagonal matrix (Lemma 3.2.2.1) and that the wedge algebras are not abelian, i.e., $g_W W \neq W$, for $W \in \mathcal{W}$ (abelian von Neumann algebras with a cyclic vector are maximally abelian).

First we assume that there is a $W \in \mathcal{W}$ such that g_W leaves a wedge $W_1 \in \mathcal{W}$ invariant. We can conclude $g_W g_{W_1} = g_{W_1} g_W \neq 1$, because of the nonabelianness of the wedge algebras ($g_{W_1} g_W W_1 \neq W_1$). There is an $L \in \mathcal{G}$ such that $g_{W_1} = g_{LW} = L g_W L^{-1}$ (transitivity) implying $g_{W_1} g_W = L g_W L^{-1} g_W \in \mathcal{P}'$. Hence, $g_W g_{W_1}$ is a nontrivial involution in $\mathcal{P}_+^\uparrow$, i.e.,

$$g_W g_{W_1} = \operatorname{diag}(1, 1, -1, -1),$$

for a suitable choice of the cordinate system. However, $g_W g_{W_1} \neq 1$ is a Poincaré transformation in the center of the invariance group of W_R. The claim easily follows in this case from the transitivity and the simplicity of $\mathcal{P}_+^\uparrow$, since every element of the Poincaré group is, up to an element of the invariance group of W_R, an element of $\mathcal{G}$ implying

$$\{K g_W g_{W_1} K^{-1} \,|\, K \in \mathcal{P}\} = \{K g_W g_{W_1} K^{-1} \,|\, K \in \mathcal{G}\}.$$

If there is not a $W \in \mathcal{W}$ such that g_W leaves a wedge $W_1 \in \mathcal{W}$ invariant, then we can conclude that the modular conjugations g_W are, for a suitable choice of the coordinate system, of the form $\operatorname{diag}(1, -1, -1, -1)$ (the transitivity excludes $g_W = -1$). However, for every boost B and every rotation R,

$$\operatorname{diag}(1, -1, -1, -1)(BR)\operatorname{diag}(1, -1, -1, -1)(BR)^{-1} = B^{-2},$$

i.e., $\mathcal{G}$ contains either a nontrivial boost B^{-2} or a nontrivial rotation R. The claim follows as in the first case. If $\mathcal{G}$ contains a nontrivial boost B^{-2}, then we can assume that B^{-2} leaves W_R invariant (choose the coordinate system suitably). To any $K \in \mathcal{P}_+^\uparrow$, there is a $G \in \mathcal{G}$ such that $K^{-1}G$ is in the invariance group of W_R, i.e., $K^{-1}G \operatorname{diag}(\pm 1, 1, 1, 1)$ commutes with B^{-2}. We can conclude $KB^{-2}K^{-1} =$

$GB^{\mp 2}G^{-1} \in \mathcal{G}$. The claim follows similarly if $\mathcal{G}$ contains a nontrivial rotation.

3.4 Geometric Action of Even Products of Modular Conjugations

In this section we will use a weakened form of the strong CGMA. Instead of (iv), we will only assume that even products of modular conjugations have a geometric action:

(iv') The adjoint action of $J_{W_1}J_{W_2}$, $W_1, W_2 \in \mathcal{W}$, leaves the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ invariant, i.e.,

$$J_{W_1}J_{W_2}\mathcal{R}(W)J_{W_2}J_{W_1} = \mathcal{R}(W_3) \quad \forall W_1, W_2, W \in \mathcal{W},$$

where $W_3 \in \mathcal{W}$ are suitable wedges.

The elements of $\mathcal{J}_e$ act on $\mathcal{W}$ as elements of the extended Poincaré group $\mathcal{DP}$ (see Theorem 3.1.3). Under the further assumption that the adjoint action of the elements of $\mathcal{J}_e$ is transitive on the net, we will show in Theorem 3.4.4 that the action is given by

$$J_{W_1}J_{W_2}\mathcal{R}(W)J_{W_2}J_{W_1} = \mathcal{R}(g_{W_1}g_{W_2}W), \quad W, W_1, W_2 \in \mathcal{W} \tag{3.4.1}$$

with

$$g_{W_R} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $g_W = \Lambda g_{W_R}\Lambda^{-1}$ for $\Lambda \in \mathcal{P}$, $W = \Lambda W_R$. Modular covariance with respect to the modular groups implies (3.4.1) (see Theorem 3.6.2.1).

Proposition 3.4.1: *Assume the strong CGMA (with (iv) replaced by (iv')) and that* $\mathcal{J}_e$ *acts transitively on the net* $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$. *The elements of* $\mathcal{J}_e$ *act as elements of* *the group $\mathcal{DP}_+^\uparrow$ which is generated by the dilatations and $\mathcal{P}_+^\uparrow$.*

Proof: Elements of $\mathcal{J}_e$ act as elements of the extended Poincaré group. Therefore, we have to show that $J_W J_{W_0}$, $W, W_0 \in \mathcal{W}$, can be written as the product of squares of elements of $\mathcal{J}_e$. There are $W_1, \ldots, W_n \in \mathcal{W}$ such that

$$J_W = J_{W_n} \cdots J_{W_1} J_{W_0} J_{W_1} \cdots J_{W_n}$$

(transitivity and uniqueness of the modular objects). Set

$$\begin{aligned} A_k &= (J_{W_k} \cdots J_{W_1} J_{W_0} J_{W_1} \cdots J_{W_k})(J_{W_{k-1}} \cdots J_{W_1} J_{W_0} J_{W_1} \cdots J_{W_{k-1}}) \\ &= (J_{W_k}(J_{W_{k-1}} \cdots J_{W_1} J_{W_0} J_{W_1} \cdots J_{W_{k-1}}))^2 \in \mathcal{J}_e^2. \end{aligned}$$

We can then rewrite

$$J_W J_{W_0} = (J_{W_n} \cdots J_{W_1} J_{W_0} J_{W_1} \cdots J_{W_n}) J_{W_0} = A_n A_{n-1} \cdots A_1$$

proving the claim.

Proposition 3.4.2: *A subgroup G of $\mathcal{DP}_+^\uparrow$ which acts transitively on $\mathcal{W}$ contains $\mathcal{P}_+^\uparrow$.*

Proof: Let $\mathcal{T} \subset \mathcal{DP}_+^\uparrow$ be the subgroup of translations. Consider the canonical homomorphism $\sigma : \mathcal{DP}_+^\uparrow \mapsto \mathcal{L}_+^\uparrow$ given by

$$\sigma(td\Lambda) = \Lambda,$$

for $t \in \mathcal{T}$, $\Lambda \in \mathcal{L}_+^\uparrow$, and dilatations d. $\sigma(G)$ acts transitively on $\mathcal{W}_0$ (we denote by $\mathcal{W}_0$ the set of wedges containing the origin in their edge) for G acts transitively on $\mathcal{W}$. This implies $\sigma(G) = \mathcal{L}_+^\uparrow$ [18, Prop. 4.2.2]. $\mathcal{L}_+^\uparrow$ is perfect, i.e., to any $\Lambda \in \mathcal{L}_+^\uparrow$ there are $\Lambda_1, \Lambda_2 \in \mathcal{L}_+^\uparrow$ such that $\Lambda = \Lambda_1^{-1}\Lambda_2^{-1}\Lambda_1\Lambda_2$. Choose $g_1, g_2 \in G$ such that $\sigma(g_1) = \Lambda_1$ and $\sigma(g_2) = \Lambda_2$.

$$\Lambda\mathcal{T} \;=\; \sigma(g_1)^{-1}\sigma(g_2)^{-1}\sigma(g_1)\sigma(g_2)\mathcal{T} \;=\; g_1^{-1}g_2^{-1}g_1g_2\mathcal{T}$$

implies that to any $\Lambda \in \mathcal{L}_+^\uparrow$ there is a suitable $t \in \mathcal{T}$ such that $t\Lambda \in G$.

It is sufficient to show that G contains a nontrivial translation (see the proof of [18, Prop. 4.2.9]). We can assume that G is not a subgroup of $\mathcal{P}_+^\uparrow$ [18, Prop. 4.2.9]. Hence, there is an element of G of the form $td\Lambda$ ($t \in \mathcal{T}$, nontrivial dilatation d, $\Lambda \in \mathcal{L}_+^\uparrow$). We have shown $t'\Lambda \in G$, for a suitable $t' \in \mathcal{T}$, and thus $t''d = td\Lambda(t'\Lambda)^{-1} \in G$ ($t'' \in \mathcal{T}$ suitable). By considering $t_0Gt_0^{-1}$ (choose $t_0 \in \mathcal{T}$ suitable such that $t_0^{-1}dt_0 = t''d$) instead of G we can assume $d \in G$ ($d = t_0t''dt_0^{-1}$). $d^{-1}g^{-1}dg$ is a nontrivial translation for every $g \in G$ which is not the product of a dilatation and an element of $\mathcal{L}_+^\uparrow$. There always exists such a $g \in G$ because of the transitivity of G.

Corollary 3.4.3: *Assume the strong CGMA (with (iv) replaced by (iv')) and that* $\mathcal{J}_e$ *acts transitively on the net* $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. *To any* $P \in \mathcal{P}_+^\uparrow$ *there is a* $K \in \mathcal{J}_e$ *such that*

$$K\mathcal{R}(W)K^{-1} \;=\; \mathcal{R}(PW) \quad \forall W \in \mathcal{W}.$$

Proof: Propositions 3.4.1 and 3.4.2

Theorem 3.4.4: *Assume the strong CGMA (with (iv) replaced by (iv')) and that* $\mathcal{J}_e$ *acts transitively on the net* $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. *The geometric action of the elements of* $\mathcal{J}_e$ *is given by (3.4.1).*

Proof: Let $\pi_m : \mathcal{J}_e \mapsto \mathcal{DP}$ be the homomorphism given by

$$K\mathcal{R}(W)K^{-1} = \mathcal{R}(\pi_m(K)W), \quad W \in \mathcal{W}.$$

The preceding corollary entails $\pi_m(\mathcal{J}_e') = \mathcal{P}_+^\uparrow$ $(\mathcal{P}_+^\uparrow = (\mathcal{P}_+^\uparrow)' \subset \pi_m(\mathcal{J}_e') \subset \mathcal{P}_+^\uparrow)$. The adjoint action of J_W, $W \in \mathcal{W}$, leaves $\mathcal{J}_e$ invariant, i.e, the characteristic subgroup $\mathcal{J}_e'$ as well. It is easy to check that this induces an automorphism α_W of $\mathcal{P}_+^\uparrow$ by

$$\alpha_W(\pi_m(K)) = \pi_m(J_W K J_W), \quad K \in \mathcal{J}_e'.$$

In order to prove (3.4.1) it is sufficient to show

$$\alpha_W(A) = g_W A g_W \quad \forall A \in \mathcal{P}_+^\uparrow$$

for this would entail

$$\begin{aligned} \pi_m(J_{W_1} J_{W_2})\pi_m(K)\pi_m(J_{W_1} J_{W_2})^* &= \pi_m(J_{W_1} J_{W_2} K J_{W_2} J_{W_1}) \\ &= \alpha_{W_1}(\alpha_{W_2}(\pi_m(K))) \\ &= g_{W_1} g_{W_2} \pi_m(K) g_{W_2} g_{W_1} \end{aligned}$$

$\forall K \in \mathcal{J}_e'$ and hence $\pi_m(J_{W_1} J_{W_2}) = g_{W_1} g_{W_2}$ (no nontrivial element of $\mathcal{DP}$ commutes with each element of $\mathcal{P}_+^\uparrow = \pi_m(\mathcal{J}_e')$). Because of the transitivity, we only have to prove

$$\alpha_{W_R}(A) = g_{W_R} A g_{W_R} \quad \forall A \in \mathcal{P}_+^\uparrow. \tag{3.4.2}$$

The automorphisms α_W leave the characteristic subgroup $\mathcal{T} \leq \mathcal{P}_+^\uparrow$ invariant. In the following we denote by $t(a,b,c,d)$, $a,b,c,d \in \mathbf{R}$, translations by $(a,b,c,d)^{\mathrm{T}}$. Set

$$\mathcal{T}_\pm = \{t \in \mathcal{T} \,|\, \alpha_{W_R}(t) = t^{\pm 1}\}.$$

J_{W_R} commutes with elements in $\mathcal{J}_e$ which act as $t(0,0,c,d)$, $c,d \in \mathbf{R}$ (uniqueness of the modular objects), i.e.,

$$\{t(0,0,c,d) \,|\, c,d \in \mathbf{R}\} \subset \mathcal{T}_+.$$

This implies $t^{-1}(R_1(\pi)tR_1(\pi)) \in \mathcal{T}_+$ for any $t \in \mathcal{T}$. Rotations about the x_1-axis leave W_R invariant showing $t^{-1}(R_1(\pi)tR_1(\pi)) \in \mathcal{T}_-$, for $t \in \mathcal{T}_-$ ($\alpha_{W_R}(R_1(\pi)) = R_1(\pi)$), and then $t^{-1}(R_1(\pi)tR_1(\pi)) \in \mathcal{T}_- \cap \mathcal{T}_+$, resp. $t = R_1(\pi)tR_1(\pi)$, for $t \in \mathcal{T}_-$. It follows

$$\mathcal{T}_- \subset \{t(a,b,0,0) \,|\, a,b \in \mathbf{R}\}.$$

The adjoint action of $J_{(0,1,0,0)^{\mathrm{T}}+W_R}J_{W_R}$ maps $\mathcal{R}(W_R)$ onto a wedge algebra which is contained in $\mathcal{R}((0,1,0,0)^{\mathrm{T}} + W_R)$ (recall (3.2.1)). Set (let $K \in \mathcal{J}_e'$ act as $t(0,1,0,0)$)

$$\begin{aligned} t_1 &= \pi_m(J_{(0,1,0,0)^{\mathrm{T}}+W_R}J_{W_R}) \\ &= \pi_m(KJ_{W_R}K^{-1}J_{W_R}) \\ &= t(0,1,0,0)\,\alpha_{W_R}(t(0,1,0,0)^{-1}) \\ &\in \mathcal{T}_-. \end{aligned}$$

t_1 maps W_R onto a wedge contained in $(0,1,0,0)^{\mathrm{T}} + W_R$. Hence, t_1 is of the form $t_1 = t(a_1,a_1,0,0)t(a_2,-a_2,0,0)$ ($a_1 \neq 0$ and $a_2 \neq 0$). Let $v_1(t)$ be the group of boosts in the direction of the x_1-axis. They leave W_R invariant implying $\alpha_{W_R}(v_1(t)) = v_1(t)$. The group $\mathcal{T}_-$ contains $t_1 \neq 1$ and is invariant with respect to boosts in the direction of the x_1-axis. It is easy to check

$$(v_1(-\frac{\ln 2}{2\pi})t_1v_1(\frac{\ln 2}{2\pi}))t_1^{-2} = t(-3a_1/2,-3a_1/2,0,0) \in \mathcal{T}_-$$

and

$$(v_1(\frac{\ln 2}{2\pi})t_1v_1(-\frac{\ln 2}{2\pi}))t_1^{-2} = t(-3a_2/2,3a_2/2,0,0) \in \mathcal{T}_-.$$

It follows

$$\{t(a,\pm a,0,0) \,|\, a \in \mathbf{R}\} \subset \mathcal{T}_-$$

(use again the invariance of $\mathcal{T}_-$ with respect to the boosts in the direction of the x_1-axis). This shows

$$\{t(a,b,0,0) \,|\, a,b \in \mathbf{R}\} = \mathcal{T}_-$$

implying

$$\begin{aligned}\alpha_{W_R}(t(a,b,c,d)) &= \alpha_{W_R}(t(a,b,0,0))\alpha_{W_R}(t(0,0,c,d)) \\ &= t(-a,-b,c,d) \\ &= g_{W_R}t(a,b,c,d)g_{W_R} \end{aligned} \qquad (3.4.3)$$

$\forall a,b,c,d \in \mathbf{R}$. α_{W_R} leaves the boosts in the direction of the x_1-axis and the rotations about the x_1-axis invariant (uniqueness of the modular objects). In order to prove (3.4.2), resp. (3.4.1), it is sufficient to show

$$\alpha_{W_R}(R_3(\varphi)) = R_3(-\varphi) \quad \forall\varphi, \qquad (3.4.4)$$

since boosts in the direction of the x_1-axis, rotations about the x_1-axis, rotations about the x_3-axis, and translations generate $\mathcal{P}_+^\uparrow$ (see [18] or [24]). (3.4.3) entails that $\alpha_{W_R}(R_3(\varphi))R_3(\varphi) \in \mathcal{P}_+^\uparrow$ commutes with all translations

$$\begin{aligned} t(a,b,c,d) &= \alpha_{W_R}(g_{W_R}t(a,b,c,d)g_{W_R}) \\ &= \alpha_{W_R}(R_3(\varphi)R_3(-\varphi)g_{W_R}t(a,b,c,d)g_{W_R}R_3(\varphi)R_3(-\varphi)) \\ &= \alpha_{W_R}(R_3(\varphi)g_{W_R}R_3(\varphi)t(a,b,c,d)R_3(-\varphi)g_{W_R}R_3(-\varphi)) \\ &= \alpha_{W_R}(R_3(\varphi))\alpha_{W_R}(g_{W_R}R_3(\varphi)t(a,b,c,d)R_3(-\varphi)g_{W_R})\alpha_{W_R}(R_3(-\varphi)) \\ &= \alpha_{W_R}(R_3(\varphi))R_3(\varphi)t(a,b,c,d)R_3(-\varphi)\alpha_{W_R}(R_3(-\varphi)), \end{aligned}$$

i.e., it is a translation itself and, hence, for suitable $t_\varphi \in \mathcal{T}$,

$$\alpha_{W_R}(R_3(\varphi)) = t_\varphi R_3(-\varphi) \quad \forall\varphi.$$

Let B be an arbitrary boost in the direction of the x_1-axis. B leaves W_R invariant, i.e.,

$$B = \alpha_{W_R}(B)$$

$$\begin{aligned} &= \alpha_{W_R}(R_3(\pi)B^{-1}R_3(\pi)) \\ &= (t_\pi R_3(\pi))B^{-1}(t_\pi R_3(\pi))^{-1} \\ &= t_\pi B t_\pi^{-1} \end{aligned}$$

implying $t_\pi = t(0,0,c_1,c_2)$, for suitable $c_1, c_2 \in \mathbf{R}$. $R_1(\pi)$ leaves W_R invariant, i.e.,

$$\begin{aligned} R_1(\pi) &= \alpha_{W_R}(R_1(\pi)) \\ &= \alpha_{W_R}(R_3(\pi)R_1(\pi)R_3(\pi)) \\ &= (t_\pi R_3(\pi))R_1(\pi)(t_\pi R_3(\pi))^{-1} \\ &= t_\pi R_1(\pi) t_\pi^{-1}. \end{aligned}$$

This proves $c_1 = c_2 = 0$, i.e.,

$$\alpha_{W_R}(R_3(\pi)) = R_3(\pi).$$

$$\begin{aligned} t_\varphi R_3(-\varphi) &= \alpha_{W_R}(R_3(\varphi)) \\ &= \alpha_{W_R}(R_3(\pi)R_3(\varphi)R_3(\pi)) \\ &= R_3(\pi)\, t_\varphi R_3(-\varphi)R_3(\pi) \\ &= R_3(\pi)\, t_\varphi R_3(\pi)R_3(-\varphi) \end{aligned}$$

proves $t_\varphi = t(a_\varphi, 0, 0, b_\varphi)$, for suitable $a_\varphi, b_\varphi \in \mathbf{R}$. As above for the rotation group $R_3(\varphi)$, we can show

$$\alpha_{W_R}(v_3(s)) = t'_s v_3(-s) \quad \forall s \in \mathbf{R}$$

for suitable $t'_s \in \mathcal{T}$ ($v_3(s)$ denotes the group of boosts in the direction of the x_3-axis).

$$\begin{aligned} (t'_s v_3(-s) t_\varphi v_3(s))(v_3(-s)R_3(-\varphi)) &= t'_s v_3(-s) t_\varphi R_3(-\varphi) \\ &= \alpha_{W_R}(v_3(s)R_3(\varphi)) \\ &= \alpha_{W_R}(R_3(\varphi)v_3(s)) \end{aligned}$$

$$\begin{aligned} &= t_\varphi R_3(-\varphi)t'_s v_3(-s) \\ &= (t_\varphi R_3(-\varphi)t'_s R_3(\varphi))(v_3(-s)R_3(-\varphi)) \end{aligned}$$

implies $t_\varphi(R_3(-\varphi)t'_s R_3(\varphi)) = t'_s(v_3(-s)t_\varphi v_3(s))$, resp.

$$(R_3(-\varphi)t'_s R_3(\varphi))t'^{-1}_s = t^{-1}_\varphi(v_3(-s)t_\varphi v_3(s)).$$

Note that the translation $(R_3(-\varphi)t'_s R_3(\varphi))t'^{-1}_s$ has trivial x_0- and x_3-components, whereas $t^{-1}_\varphi(v_3(-s)t_\varphi v_3(s))$ has trivial x_1- and x_2-components. We can conclude

$$(R_3(-\varphi)t'_s R_3(\varphi))t'^{-1}_s = t^{-1}_\varphi(v_3(-s)t_\varphi v_3(s)) = 1,$$

resp. $t_\varphi = v_3(-s)t_\varphi v_3(s)$ $\forall s \in \mathbf{R}$, implying $a_\varphi = b_\varphi = 0$, i.e., (3.4.4) is proven.

3.5 Construction of the Poincaré Group

In order to be able to apply Moore's cohomology theory we assume that the Hilbert space $\mathcal{H}$ is separable.

In this section we consider an order-preserving bijection $W \mapsto \mathcal{R}(W)$ and assume (3.4.1), which is equivalent to

$$J_{W_1}\mathcal{R}(W_2)J_{W_1} = J_{W_1}J_{W_2}\mathcal{R}(W_2)'J_{W_2}J_{W_1} = \mathcal{R}(g_{W_1}W_2')', \quad W_1, W_2 \in \mathcal{W}.$$

Note that (3.4.1) can be derived from Theorem 3.4.4. We will show that (3.4.1) implies the existence of a strongly continuous representation of the translation group. If we use, in addition,

(v) Ω is cyclic for $\mathcal{R}(W_1)\cap\mathcal{R}(W_2) \subset \mathcal{R}(W)$, for $\emptyset \neq W_1\cap W_2 \subset W$ and $W_1'\cap W_2' \neq \emptyset$,

then we can even prove the existence of a strongly continuous representation U of a covering group $\widetilde{\mathcal{P}}_+$ of $\mathcal{P}_+$ such that $U(\widetilde{\mathcal{P}}_+) = \mathcal{J}$. If locality is fulfilled, then

$$J_{W_0}\mathcal{R}(W)J_{W_0} = \mathcal{R}(g_{W_0}W), \quad W, W_0 \in \mathcal{W},$$

and we even obtain a representation of $\mathcal{P}_+$. Under certain conditions it was shown in [38] that quantum field theories satisfy our assumption (v). Furthermore, we use this relation only for wedges associated to modular intersections (see below). This condition was also implicitly assumed in [8], where Borchers derived the existence of a strongly continuous representation of the Poincaré group from modular covariance. The claim follows too if we replace (v) by the assumption

(v') Ω is cyclic for $\cap_{i\in I}\mathcal{R}(W_i)$, whenever $\cap_{i\in I}W_i$ contains arbitrarily large double cones.

(v') is a consequence of isotony and the positivity of energy if we consider a local net of von Neumann algebras.

3.5.1 The Translation Group

Lemma 3.5.1.1: *Let W be a wedge and $x \in \mathbf{R}^4$ be a vector such that $W + x \subset W$. Denote by J_t the modular conjugation of $\mathcal{R}(W + tx)$, $t \in \mathbf{R}$. $\{J_t\}$ converges strongly to J_0 for $t \to 0$.*

Proof: Let $\{t_n\}$ be a decreasing sequence in $\mathbf{R}$ which converges to 0. $\{J_{t_n}\}$ converges strongly to the modular conjugation J of

$$(\cup_n \mathcal{R}(W + t_n x))'' = \mathcal{R} \subset \mathcal{R}(W)$$

[17, Corollary A.2]. We will show $J = J_0$. Consider an arbitrary $A \in \mathcal{R}(W + tx)$, $t > 0$.

$$J_0 J_{t_n} A J_{t_n} J_0 \in \mathcal{R}(W + (t - 2t_n)x)$$

entails $J_0 J A J J_0 \in \mathcal{R}$, resp. $J_0 J \mathcal{R} J J_0 \subset \mathcal{R}$.

$$\mathcal{R} \subset \mathcal{R}(W) = J_0 \mathcal{R}(W)' J_0 \subset J_0 \mathcal{R}' J_0 = J_0 J \mathcal{R} J J_0 \subset \mathcal{R}$$

proves $\mathcal{R} = \mathcal{R}(W)$. Similarly, we can show the strong convergence of $\{J_{t_n}\}$ to J_0 if $\{t_n\}$ is increasing to 0.

Let $\{t_n\}$ now be a sequence in $\mathbf{R}$ which converges to 0. There is a subsequence $\{t_{k_n}\}$ such that either $\{t_{k_n}\}$ is decreasing to 0 or such that $\{t_{k_n}\}$ is increasing to 0. In both cases we can conclude that $\{J_{t_{k_n}}\}$ converges strongly to J_0, i.e., J_0 is, in the weak topology, the only accumulation point of $\{J_t\}$ and $\{J_t\}$ converges weakly, hence strongly, to J_0 $(t \to 0)$.

Lemma 3.5.1.2: *Let W be a wedge and $x \in \mathbf{R}^4$ be a vector. Denote by J_t the modular conjugation of $\mathcal{R}(W + tx)$, $t \in \mathbf{R}$. If $t \mapsto J_t$ is strongly continuous, then $U(t) = J_t J_0$ defines a strongly continuous one-dimensional group.*

Proof: see the proof of [18, Lemma 4.3.4]

Theorem 3.5.1.3: *We can define a strongly continuous representation of the translations by*

$$V(x_0, x_1, x_2, x_3) = V_0(x_0)V_1(x_1)V_2(x_2)V_3(x_3),$$

$$V_0(2x_0) = J_{(x_0,0,0,0)^{\mathrm{T}}+W_R} J_{W_R}, \quad V_1(2x_1) = J_{(0,x_1,0,0)^{\mathrm{T}}+W_R} J_{W_R}$$

and

$$V_2(2x_2) = J_{(0,0,x_2,0)^{\mathrm{T}}+W_2^{(0)}} J_{W_2^{(0)}}, \quad V_3(2x_3) = J_{(0,0,0,x_3)^{\mathrm{T}}+W_3^{(0)}} J_{W_3^{(0)}}.$$

Proof: $V_i(t)$, $i = 1,2,3$, are strongly continuous groups by Lemma 3.5.1.1 and Lemma 3.5.1.2. $U_\pm(x_0) = J_{(x_0/4, \pm x_0/4,0,0)^{\mathrm{T}}+W_R} J_{W_R}$ acts as a translation by $(x_0/2, \pm x_0/2, 0, 0)^{\mathrm{T}}$ implying

$$J_{(x_0,0,0,0)^{\mathrm{T}}+W_R} = U_+(x_0)U_-(x_0)J_{W_R}U_-(-x_0)U_+(-x_0).$$

Therefore, $V_0(t)$ is strongly continuous (Lemma 3.5.1.1) and it is also a group according to Lemma 3.5.1.2. V defines a representation (see [18, 4.3]).

Lemma 3.5.1.4:

$$J_W V(x) J_W \;=\; V(g_W x) \quad \forall x \in \mathbf{R}^4$$

for any wedge W which contains the origin in its edge.

Proof: Set

$$Z(t) \;=\; V(-g_W tx) J_W V(tx) J_W$$

($x \in \mathbf{R}^4$ arbitrary but fixed). $Z(t)$ is a one-dimensional subgroup of the center of $\mathcal{J}$:

$$Z(t)\mathcal{R}(W_0)Z(t)^* \;=\; \mathcal{R}(W_0) \quad \forall W_0 \in \mathcal{W},\; \forall t \in \mathbf{R},$$

according to (3.4.1) ($V(tx) \in \mathcal{J}_e$ implies $J_W V(tx) J_W \in \mathcal{J}_e$), i.e., $Z(t)$, $t \in \mathbf{R}$, commutes with J_{W_0} $\forall W_0 \in \mathcal{W}$ and, furthermore,

$$\begin{aligned} Z(t+s) &= V(-g_W(s+t)x) J_W V((s+t)x) J_W \\ &= V(-g_W sx) V(-g_W tx) J_W V(tx) J_W J_W V(sx) J_W \\ &= V(-g_W sx) Z(t) J_W V(sx) J_W \\ &= Z(t)Z(s). \end{aligned}$$

$I = J_{W_R} J_{W_2^{(0)}} J_{W_3^{(0)}}$ acts as an inversion. However, I does not necessarily leave the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ invariant. Note ($I J_{x+W} \in \mathcal{J}_e$)

$$I J_{x+W} I^* \;=\; (I J_{x+W}) J_{x+W} (J_{x+W} I^*) \;=\; J_{-(x+W)'} \;=\; J_{-(x-W)} \;=\; J_{-x+W}$$

for $x \in \mathbf{R}^4$ and wedges W which contain the origin in their edge. $IV(x)I^* = V(-x)$ follows now easily from the definition of $V(x)$ implying

$$Z(t) \;=\; IZ(t)I^* \;=\; Z(-t).$$

This proves $Z(2t) = 1$.

3.5.2 The Lorentz Group

Throughout this section we assume (v). We denote by $v_1(t)$ the boosts in the direction of the x_1-axis, i.e.,

$$v_1(t) = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t & 0 & 0 \\ -\sinh 2\pi t & \cosh 2\pi t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We consider a one-dimensional subgroup $V : \mathbf{R} \mapsto \mathcal{P}_+^\uparrow$ of the stabilizer group of $(1,1,0,0)^{\mathrm{T}}$ satisfying

$$v_1(t)V(s)v_1(-t) = V(e^{-2\pi t}s) \quad \forall s, t \in \mathbf{R} \tag{3.5.2.1}$$

(see (2.5) in [8] or [44]). To every $v_1(t)$ we choose a $B(t) \in \mathcal{J}_e$ which acts as $v_1(t)$.

Lemma 3.5.2.1: *Set* $U(a) = J_{V(a/2)W_R}J_{W_R}$. $U(a)$ *acts as* $V(a)$. *Furthermore,*

$$U(a)^n = U(na) \quad \forall a \in \mathbf{R},\ n \in \mathbf{Z}.$$

Proof: The adjoint action of g_{W_R} inverts the elements $V(a)$ (see (2.5) in [8]). It follows now that $U(a)$ acts as

$$g_{V(a/2)W_R}g_{W_R} = V(a/2)g_{W_R}V(-a/2)g_{W_R} = V(a).$$

Since $U(a)$ and $U(a/2)$ commute ($U(a)$ differs from $U(a/2)^2$ by a central element of $\mathcal{J}$), we can conclude, for $n \in \mathbf{Z}$,

$$\begin{aligned} U(a) &= U(a/2)^n U(a) U(a/2)^{-n} \\ &= U(a/2)^n J_{V(a/2)W_R} J_{W_R} U(a/2)^{-n} \\ &= J_{V((n+1)a/2)W_R} J_{V(na/2)W_R}. \end{aligned}$$

Now it follows easily $U(a)^{-1} = U(-a)$ and, for $n \in \mathbf{N}$,

$$\begin{aligned} U(a)^n &= (J_{V(na/2)W_R} J_{V((n-1)a/2)W_R}) \cdot \ldots \cdot (J_{V(a)W_R} J_{V(a/2)W_R})(J_{V(a/2)W_R} J_{W_R}) \\ &= J_{V(na/2)W_R} J_{W_R} \\ &= U(na). \end{aligned}$$

Lemma 3.5.2.2: *$U(a)$ is a group.*

Proof: Choose arbitrary $a, b \in \mathbf{R}$. $[U(a), U(b)] = U(a)^{-1}U(b)^{-1}U(a)U(b)$ is a central element in $\mathcal{J}$ and

$$B(\frac{\ln 2}{2\pi})^{-1}\, U(a)\, B(\frac{\ln 2}{2\pi}) \;=\; J_{v_1(-\frac{\ln 2}{2\pi})V(a/2)W_R} J_{v_1(-\frac{\ln 2}{2\pi})W_R} \overset{(3.5.2.1)}{=} U(2a).$$

This entails

$$[U(a), U(b)] \;=\; B(\frac{\ln 2}{2\pi})^{-1}\, [U(a), U(b)]\, B(\frac{\ln 2}{2\pi}) \;=\; [U(2a), U(2b)].$$

Elements of $\mathcal{J}$ which act in the same manner differ only by a central element, i.e., they commute. It follows now easily from Lemma 3.5.2.1

$$\begin{aligned} [U(a), U(b)] &= [U(2a), U(2b)] \\ &= U(2a)^{-1}U(2b)^{-1}U(2a)U(2b) \\ &= U(a)^{-2}(U(2b)^{-1}U(a)U(2b))^2 \\ &= (U(a)^{-1}U(2b)^{-1}U(a)U(2b))^2 \\ &= ((U(a)^{-1}U(b)^{-1}U(a))^2 U(b)^2)^2 \\ &= (U(a)^{-1}U(b)^{-1}U(a)U(b))^4 \\ &= [U(a), U(b)]^4, \end{aligned}$$

i.e., $[U(a), U(b)]^3 = 1$. By using similar arguments we can even conclude

$$\begin{aligned}[U(a), U(b)] &= U(a)^{-1}U(b)^{-1}U(a)U(b) \\ &= U(a/3)^{-3}(U(b)^{-1}U(a/3)U(b))^3 \\ &= (U(a/3)^{-1}(U(b)^{-1}U(a/3)U(b)))^3 \\ &= [U(a/3), U(b)]^3 \\ &= 1.\end{aligned}$$

Now we can show that $U(a)$ is a group:

$$\begin{aligned}U(a)U(b) &= (U(b/2)U(a)U(b/2)^{-1})U(b) \\ &= J_{V(b/2)V(a/2)W_R}J_{V(b/2)W_R}J_{V(b/2)W_R}J_{W_R} \\ &= J_{V(b/2)V(a/2)W_R}J_{W_R} \\ &= U(a+b)\end{aligned}$$

It will take several lemmas to show that the set $\mathcal{U}$ of weak limit points of sequences $\{U(a_n)\}$, $a_n \to 0$, is a compact abelian group of unitary operators.

Lemma 3.5.2.3: *$A \in \mathcal{U}$ entails $A^* \in \mathcal{U}$. $A, B \in \mathcal{U}$ implies $AB \in \mathcal{U}$. Furthermore, $\mathcal{U}$ is weakly closed.*

Proof: $U(t_n) \to A$ implies $U(-t_n) \to A^*$ proving the first claim. If $A, B \in \mathcal{U}$, then there are sequences $\{s_n\}$, $\{t_n\}$ in $\mathbf{R}$ converging to 0 such that $U(s_n) \to A$, $U(t_n) \to B$. Let $d(\cdot,\cdot)$ be a metric on the unit ball of bounded operators on $\mathcal{H}$ which induces the weak topology [38, Proposition 2.7]. There is a subsequence $\{t_{k_n}\}$ of $\{t_n\}$ such that

$$d(U(s_n)U(t_{k_n}), U(s_n)B) \overset{n\to\infty}{\longrightarrow} 0,$$

i.e.,

$$d(U(s_n + t_{k_n}), AB) \;\le\; d(U(s_n)U(t_{k_n}), U(s_n)B) + d(U(s_n)B, AB) \overset{n\to\infty}{\longrightarrow} 0.$$

We have proven that $A, B \in \mathcal{U}$ implies $AB \in \mathcal{U}$. With the help of the triangle inequality it is also easy to show that $\mathcal{U}$ is weakly closed.

Lemma 3.5.2.4: *If $\mathcal{U}$ contains operators which are not unitary, then $\mathcal{U}$ contains a projection $P \neq 1$.*

Proof: If $A \in \mathcal{U}$ is not unitary, then either $A^*A \neq 1$ or $AA^* \neq 1$, i.e., $\mathcal{U}$ contains a positive operator $B \neq 1$. Since $0 \le B \le 1$, $\{B^n\}$ converges weakly towards a projection $P \neq 1$.

Lemma 3.5.2.5: *Set $\mathcal{R}_a = \mathcal{R}(V(a)W_R)$. If $a_1 \le a_2 \le a_3$, then*

$$\mathcal{R}_{a_1} \cap \mathcal{R}_{a_3} \;\subset\; \mathcal{R}_{a_2}.$$

Proof: We can assume $a_1 = 0$. The proof of Lemma 2.6 in [8] implies, for $t \ge 0$,

$$\begin{aligned} W_R \cap V(a_3)W_R \quad &\subset \quad v_1(t)(W_R \cap V(a_3)W_R) \\ &\overset{(3.5.2.1)}{=} \quad W_R \cap V(e^{-2\pi t}a_3)W_R \\ &\subset \quad V(e^{-2\pi t}a_3)W_R \end{aligned}$$

proving the claim.

In the following we will denote by S the strip $\{z \in \mathbf{C} \,|\, 0 < \mathrm{Im} z < \frac{1}{2}\}$.

Lemma 3.5.2.6: *Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras for which Ω is cyclic and separating, i.e. the corresponding modular objects $\Delta_{\mathcal{N}}, J_{\mathcal{N}}, \Delta_{\mathcal{M}}, J_{\mathcal{M}}$ are well-defined. The function $t \in \mathbf{R} \mapsto (\varphi, \Delta_{\mathcal{N}}^{it} J_{\mathcal{M}} J_{\mathcal{N}} \Delta_{\mathcal{N}}^{-it} \psi)$, $\varphi, \psi \in \mathcal{H}$, has a continuous extension $f(z)$ to $\overline{S}$ which is bounded by $\| \varphi \| \, \| \psi \|$, analytic on S, and such that*

$$f(t + i/4) \ = \| \overline{\Delta_{\mathcal{M}}^{1/4} \Delta_{\mathcal{N}}^{-1/4-it}} \varphi \|^2, \quad \textit{for} \quad t \in \mathbf{R},\ \varphi = \psi.$$

The operator $\Delta_{\mathcal{M}}^{1/4} \Delta_{\mathcal{N}}^{-1/4}$ is defined on the domain of $\Delta_{\mathcal{N}}^{-1/4}$ and the norm of its closure is bounded by 1.

Proof: First we want to show that $t \in \mathbf{R} \mapsto (\varphi, \Delta_{\mathcal{N}}^{it} J_{\mathcal{M}} J_{\mathcal{N}} \Delta_{\mathcal{N}}^{-it} \psi)$ has a continuous extension to $\overline{S}$ which is bounded by $\| \varphi \| \, \| \psi \|$ and analytic on S. It is sufficient to prove this for $\varphi \in \mathcal{N}'\Omega$ and $\psi \in \mathcal{N}\Omega$. In this case, $f(z)$ is analytic on S, continuous on $\overline{S}$, and bounded by $\sup_{0 \leq t \leq 1/2} \| \Delta_{\mathcal{N}}^{-t} \varphi \| \, \| \Delta_{\mathcal{N}}^{t} \psi \|$. Note

$$J_{\mathcal{N}} J_{\mathcal{M}} \mathcal{N} J_{\mathcal{M}} J_{\mathcal{N}} \ \subset \ J_{\mathcal{N}} J_{\mathcal{M}} \mathcal{M} J_{\mathcal{M}} J_{\mathcal{N}} \ = \ J_{\mathcal{N}} \mathcal{M}' J_{\mathcal{N}} \subset J_{\mathcal{N}} \mathcal{N}' J_{\mathcal{N}} \ = \ \mathcal{N},$$

which implies, for any $N \in \mathcal{N}$,

$$\begin{aligned} J_{\mathcal{N}} J_{\mathcal{M}} (J_{\mathcal{N}} \Delta_{\mathcal{N}}^{1/2}) N \Omega &= J_{\mathcal{N}} J_{\mathcal{M}} N^* \Omega \\ &= J_{\mathcal{N}} J_{\mathcal{M}} N^* J_{\mathcal{M}} J_{\mathcal{N}} \Omega \\ &= (J_{\mathcal{N}} \Delta_{\mathcal{N}}^{1/2}) J_{\mathcal{N}} J_{\mathcal{M}} N \Omega, \end{aligned}$$

resp. $(t \in \mathbf{R})$

$$\begin{aligned} f(t + i/2) &= (\Delta_{\mathcal{N}}^{-it-1/2} \varphi, J_{\mathcal{M}} J_{\mathcal{N}} \Delta_{\mathcal{N}}^{-it+1/2} \psi) \\ &= (J_{\mathcal{N}} (J_{\mathcal{N}} \Delta_{\mathcal{N}}^{1/2})^* \Delta_{\mathcal{N}}^{-it} \varphi, J_{\mathcal{M}} (J_{\mathcal{N}} \Delta_{\mathcal{N}}^{1/2}) \Delta_{\mathcal{N}}^{-it} \psi) \\ &= \overline{((J_{\mathcal{N}} \Delta_{\mathcal{N}}^{1/2})^* \Delta_{\mathcal{N}}^{-it} \varphi, J_{\mathcal{N}} J_{\mathcal{M}} (J_{\mathcal{N}} \Delta_{\mathcal{N}}^{1/2}) \Delta_{\mathcal{N}}^{-it} \psi)} \\ &= \overline{((J_{\mathcal{N}} \Delta_{\mathcal{N}}^{1/2})^* \Delta_{\mathcal{N}}^{-it} \varphi, (J_{\mathcal{N}} \Delta_{\mathcal{N}}^{1/2}) J_{\mathcal{N}} J_{\mathcal{M}} \Delta_{\mathcal{N}}^{-it} \psi)} \\ &= (\Delta_{\mathcal{N}}^{-it} \varphi, J_{\mathcal{N}} J_{\mathcal{M}} \Delta_{\mathcal{N}}^{-it} \psi). \end{aligned}$$

This shows, in particular, that f is bounded by $\| \varphi \| \| \psi \|$ (maximum principle).

In order to prove the last claim we consider $t \in \mathbf{R} \mapsto (\Delta_{\mathcal{M}}^{it}\varphi, \Delta_{\mathcal{N}}^{it}\psi)$, $\varphi, \psi \in \mathcal{H}$. This function has a continuous extension $g(z)$ to $\overline{S}$ which is analytic on S and bounded by $\| \varphi \| \| \psi \|$ such that $g(t + i/2) = (\Delta_{\mathcal{M}}^{it}\varphi, J_{\mathcal{M}} J_{\mathcal{N}} \Delta_{\mathcal{N}}^{it}\psi)$ for $t \in \mathbf{R}$ (see the proof of [16, Lemma 2.4]). It is easy to show $g(it) = (\Delta_{\mathcal{M}}^{t}\varphi, \Delta_{\mathcal{N}}^{-t}\psi)$, $0 \leq t \leq 1/2$, for $\varphi \in D(\Delta_{\mathcal{M}}^{t})$ and $\psi \in D(\Delta_{\mathcal{N}}^{-t})$. This entails that $\Delta_{\mathcal{M}}^{t}\Delta_{\mathcal{N}}^{-t} = (\Delta_{\mathcal{M}}^{t})^{*}\Delta_{\mathcal{N}}^{-t}$ is defined on $D(\Delta_{\mathcal{N}}^{-t})$ and its closure is norm-bounded by 1 (recall that g is bounded by $\| \varphi \| \| \psi \|$). Note $\overline{\Delta_{\mathcal{M}}^{1/2}\Delta_{\mathcal{N}}^{-1/2}} = J_{\mathcal{M}} J_{\mathcal{N}}$ (consider $g(i/2)$).

For arbitrary $\varphi = \psi \in \mathcal{H}$ which is analytic with respect to $\Delta_{\mathcal{N}}^{it}$ we can show now

$$\begin{aligned} f(t + i/4) &= (\Delta_{\mathcal{N}}^{-it-1/4}\varphi, J_{\mathcal{M}} J_{\mathcal{N}} \Delta_{\mathcal{N}}^{-it+1/4}\varphi) \\ &= (\Delta_{\mathcal{N}}^{-it-1/4}\varphi, \overline{\Delta_{\mathcal{M}}^{1/2}\Delta_{\mathcal{N}}^{-1/2}} \Delta_{\mathcal{N}}^{-it+1/4}\varphi) \\ &= (\Delta_{\mathcal{N}}^{-it-1/4}\varphi, \Delta_{\mathcal{M}}^{1/2}\Delta_{\mathcal{N}}^{-1/2} \Delta_{\mathcal{N}}^{-it+1/4}\varphi) \\ &= \| \Delta_{\mathcal{M}}^{1/4}\Delta_{\mathcal{N}}^{-it-1/4}\varphi \|^{2} . \end{aligned}$$

In the next lemma we will often use the following fact [17, Corollary A.2]. Let $\{\mathcal{M}_n\}$ be an increasing sequence of von Neumann algebras for which Ω is a cyclic and separating vector. If Ω is separating for $(\cup_n \mathcal{M}_n)''$, then the modular operators (resp. modular conjugations) of $\{(\mathcal{M}_n, \Omega)\}$ converge in the strong resolvent sense to the modular operator (resp. strongly to the modular conjugation) of $((\cup_n \mathcal{M}_n)'', \Omega)$.

Lemma 3.5.2.7: *$\mathcal{U}$ is a compact abelian group of unitary operators.*

Proof: We assume that $\mathcal{U}$ contains a projection $\lim_{t_n \to 0} U(t_n) = P \neq 1$ and derive a contradiction (see Lemma 3.5.2.4). By considering a suitable subsequence, we can assume that $\{t_n\}$ is decreasing or increasing to 0. Let $\Delta_{a,b}$, $J_{a,b}$ be the modular objects of $(\mathcal{R}_a \cap \mathcal{R}_b, \Omega)$ for $a, b \in \mathbf{R}$. $\{\mathcal{R}_{2t_n} \cap \mathcal{R}_0\}$ is an increasing sequence, according to Lemma 3.5.2.5. Therefore, $\{\Delta^{it}_{2t_n,0}\}$ converge strongly to the modular unitaries Δ^{it} of $((\cup_n(\mathcal{R}_{2t_n} \cap \mathcal{R}_0))'', \Omega)$.

Choose $\varphi \in \mathcal{H}\backslash\{0\}$ such that $P\varphi = 0$. Let Δ_a, J_a be the modular objects of $\mathcal{R}_a$ and let $f_{\varphi n}(z)$ be the continuous extension of $(\varphi, \Delta^{it}_{2t_n,0} J_{t_n/2} J_{2t_n,0} \Delta^{-it}_{2t_n,0}\varphi)$ to $\overline{S}$ which is analytic in S (see Lemma 3.5.2.6 and note $\mathcal{R}_{2t_n} \cap \mathcal{R}_0 \subset \mathcal{R}_{t_n/2}$). We want to show that $\{f_{\varphi n}(z)\}_{n \in \mathbf{N}}$ converges pointwise on $\overline{S}$ to 0. Later we will find a contradiction by proving that this is impossible. $\{\mathcal{R}_{t_n} \cap \mathcal{R}_{-t_n}\}$ is an increasing sequence, according to Lemma 3.5.2.5. Therefore, $\{\Delta^{it}_{t_n,-t_n}\}$ (resp. $\{J_{t_n,-t_n}\}$) converge strongly to the modular unitaries (resp. to the modular conjugation) of $((\cup_n(\mathcal{R}_{t_n} \cap \mathcal{R}_{-t_n}))'', \Omega)$ (note that Lemma 3.5.2.5 implies $\cup_n(\mathcal{R}_{t_n} \cap \mathcal{R}_{-t_n}) \subset \mathcal{R}_0$). $U(-t_n)(\mathcal{R}_{2t_n} \cap \mathcal{R}_0)U(t_n) = \mathcal{R}_{t_n} \cap \mathcal{R}_{-t_n}$ implies

$$\begin{aligned} f_{\varphi n}(t) &= (\varphi, \Delta^{it}_{2t_n,0} U(t_n) J_0 J_{2t_n,0} \Delta^{-it}_{2t_n,0}\varphi) \\ &= (U(-t_n)\varphi, \Delta^{it}_{t_n,-t_n} J_0 J_{2t_n,0} \Delta^{-it}_{2t_n,0}\varphi), \end{aligned}$$

i.e., $\{f_{\varphi n}(t)\}$ converges to 0 for $t \in \mathbf{R}$ ($\{U(-t_n)\}$ converges weakly to $P^* = P$ and $\{\Delta^{it}_{t_n,-t_n} J_0 J_{2t_n,0} \Delta^{-it}_{2t_n,0}\varphi\}$ converges strongly). Similarly, we can prove that $\{f_{\varphi n}(t + i/2)\}$ converges to 0 for $t \in \mathbf{R}$. In order to show that $\{f_{\varphi n}(z)\}$ converges to 0 for all $z \in \overline{S}$ it is sufficient to consider continuous functions $\{g_n(z)\}$ which are defined on $\{z \in \mathbf{C} \,|\, |z| \leq 1\}\backslash\{z_1, z_2\}$ for fixed z_1, z_2, $|z_1| = |z_2| = 1$, bounded by 1, and analytic on $\{z \in \mathbf{C} \,|\, |z| < 1\}$ (compose $\{f_{\varphi n}(z)\}$ with a conformal map which maps S onto $\{z \in \mathbf{C} \,|\, |z| < 1\}$). With the help of the Poisson kernel and the bounded convergence theorem we can show $g_n(z) \to 0$ for $|z| < 1$, resp. $f_{\varphi n}(z) \to 0$ for $z \in \overline{S}$.

Now we will find a contradiction. Ω is separating for $(\cup_n \mathcal{R}_{t_n/2})''$, since Ω is cyclic for

$$\begin{aligned} (\cup_n \mathcal{R}_{t_n/2})' &= \cap_n \mathcal{R}'_{t_n/2} \supset \mathcal{R}'_0 \cap (\cap_n \mathcal{R}'_{t_n/2}) \\ &= J_0(\mathcal{R}_0 \cap (\cap_n J_0 J_{t_n/2} \mathcal{R}_{t_n/2} J_{t_n/2} J_0))J_0 = J_0(\mathcal{R}_0 \cap (\cap_n \mathcal{R}_{-t_n/2}))J_0 \\ &= J_0(\mathcal{R}_0 \cap \mathcal{R}_{-t_1/2})J_0 \end{aligned}$$

(recall Lemma 3.5.2.5 and that $\{t_n\}$ is monotone). Therefore, the modular group Δ'^{it} of $((\cup_n \mathcal{R}_{t_n/2})'', \Omega)$ is well-defined. Lemma 3.5.2.6 implies

$$\begin{aligned} 0 &= \lim_n f_{\varphi n}(t + i/4) \\ &= \lim_n \| \overline{\Delta_{t_n/2}^{1/4} \Delta_{2t_n,0}^{-1/4-it}} \varphi \|^2 \\ &\geq \lim_n \| \overline{\Delta'^{1/4} \Delta_{t_n/2}^{-1/4}}\, \overline{\Delta_{t_n/2}^{1/4} \Delta_{2t_n,0}^{-1/4-it}} \varphi \|^2 \\ &= \lim_n \| \overline{\Delta'^{1/4} \Delta_{2t_n,0}^{-1/4-it}} \varphi \|^2 \\ &= \| \overline{\Delta'^{1/4} \Delta^{-1/4-it}} \varphi \|^2 \\ &\geq 0. \end{aligned}$$

The last equality follows from the fact that $\Delta^{-1/4}$ is the strong graph limit of $\{\Delta_{2t_n,0}^{-1/4}\}$. We only have to show $\| \overline{\Delta'^{1/4} \Delta^{-1/4-it}} \varphi \| \neq 0$, for a $t \in \mathbf{R}$. However, the function $(\Delta'^{it}\varphi, \Delta^{it}\varphi)$ has a continuous extension to $\overline{S}$ which is analytic on S (see the proof of Lemma 3.5.2.6) and which is not identical 0, since it is equal to $\| \varphi \|^2$ at $t = 0$. Furthermore, it agrees with

$$(\varphi, \overline{\Delta'^{-it+1/4} \Delta^{it-1/4}} \varphi) = (\Delta'^{it}\varphi, \overline{\Delta'^{1/4} \Delta^{it-1/4}} \varphi)$$

at $t + i/4$ $(t \in \mathbf{R})$. This entails $(\Delta'^{it}\varphi, \overline{\Delta'^{1/4} \Delta^{it-1/4}} \varphi) \neq 0$, for a $t \in \mathbf{R}$ (zeros of nonconstant analytic functions are isolated), proving the claim.

$\mathcal{U}$ is weakly closed, i.e., $\mathcal{U}$ is compact with respect to the weak, resp. strong, topology. Furthermore, $\mathcal{U}$ is abelian (Lemma 3.5.2.2).

Proposition 3.5.2.8: *$U(a)$ is a strongly continuous group.*

Proof: We have to show that $\mathcal{U}$ only contains the identity. Since any representation of a compact group is a direct sum of finite-dimensional irreducible representations, which are one-dimensional for abelian groups, there is a countable set I, subspaces $\mathcal{H}_i \subset \mathcal{H}$, and pairwise different characters $\chi_i : \mathcal{U} \mapsto \{z \in \mathbf{C} \mid |z| = 1\}$ such that

$$A\varphi = \chi_i(A)\varphi \quad \forall A \in \mathcal{U}, \varphi \in \mathcal{H}_i$$

$(i \in I)$. Since the adjoint action of elements of $\mathcal{J}_e$ which act as boosts in the direction of the x_1-axis leaves $\mathcal{U}$ invariant, it is not too difficult to show that to any $t \in \mathbf{R}$ and $i \in I$ there is an $i' \in I$ such that $B(t)\mathcal{H}_i = \mathcal{H}_{i'}$ (i' is fixed by $\chi_{i'}(A) = \chi_i(B(t)^{-1}AB(t))$ for $A \in \mathcal{U}$).

Choose arbitrary $i_0 \in I$ and $A \in \mathcal{U}$. It is sufficient to show that A acts trivially on $\mathcal{H}_{i_0}$. There are $t_1 \neq t_2$ such that $B(t_1)\mathcal{H}_{i_0} = B(t_2)\mathcal{H}_{i_0}$ ($\mathbf{R}$ is uncountable but I is countable), i.e., such that

$$B(t_1)^{-1}B(t_2)\mathcal{H}_{i_0} = \mathcal{H}_{i_0}.$$

Set $B = B(t_1)^{-1}B(t_2)$. B differs from $B(t_2 - t_1)$ only by a central element of $\mathcal{J}$. There is a sequence $\{s_n\}$ which converges to 0 such that $A = \lim_n U(s_n)$. Let $\{s'_n\}$ be the sequence which is determined by

$$\begin{aligned} U(s_n) &= U((e^{2\pi(t_1-t_2)} - 1)s'_n) \\ &= U(e^{2\pi(t_1-t_2)}s'_n)U(-s'_n) \\ &= BU(s'_n)B^{-1}U(-s'_n). \end{aligned}$$

The sequence $\{U(s'_n)\}$ converges too (adjoin $\{U(s_n)\}$ by an element of $\mathcal{J}_e$ which acts as a suitable boost in order to obtain $\{U(s'_n)\}$). We will call the limit of this sequence

A'. Note

$$A = BA'B^{-1}A'^{*}$$

and that A' acts as a multiple of the identity on $\mathcal{H}_{i_0}$ implying

$$A|_{\mathcal{H}_{i_0}} = B|_{\mathcal{H}_{i_0}} A'|_{\mathcal{H}_{i_0}} B^{-1}|_{\mathcal{H}_{i_0}} A'^{*}|_{\mathcal{H}_{i_0}} = 1|_{\mathcal{H}_{i_0}}.$$

This proves the claim.

We will denote by $\mathcal{W}_0$ the set of wedges containing the origin in their edge and by $\mathcal{J}_0$ the group consisting of the even products of modular conjugations corresponding to wedges in $\mathcal{W}_0$.

Lemma 3.5.2.9: *$W \in \mathcal{W}_0 \mapsto J_W$ is continuous if we equip $\mathcal{W}_0$ with the same topology as in [18] (see Section 3.2).*

Proof: Since $W \mapsto \mathcal{R}(W)$ is a bijection, we can define a homomorphism $\varphi_0 : \mathcal{J}_0 \mapsto \mathcal{L}_+^\uparrow$ by

$$V\mathcal{R}(W)V^* = \mathcal{R}(\varphi_0(V)W) \quad \forall W \in \mathcal{W}, V \in \mathcal{J}_0.$$

According to Proposition 3.5.2.8, there exist strongly continuous one-parameter subgroups $V_1(t), \ldots, V_6(t)$ of $\mathcal{J}_0$ such that the generators of $\varphi_0(V_1(t)), \ldots, \varphi_0(V_6(t))$ form a basis of the Lie algebra of $\mathcal{L}_+^\uparrow$ ($\mathcal{L}_+^\uparrow$ is simple).

If $\{W_n \in \mathcal{W}\}$ converges to $W \in \mathcal{W}$, then there is a sequence $\{L_n \in \mathcal{L}_+^\uparrow\}$ which converges to the identity such that $W_n = L_n W$. Hence, there are sequences $\{t_{1_n}\}, \ldots, \{t_{6_n}\}$ such that $t_{k_n} \to 0$ for $k = 1, \ldots, 6$ and $L_n = \varphi_0(V_1(t_{1_n})) \cdot \ldots \cdot \varphi_0(V_6(t_{6_n}))$.

$$J_{W_n} = J_{L_n W} = (V_1(t_{1_n}) \cdot \ldots \cdot V_6(t_{6_n}))J_W(V_1(t_{1_n}) \cdot \ldots \cdot V_6(t_{6_n}))^*$$

implies the strong convergence of $J_{W_n} \to J_W$.

Of course, we could even prove the continuity for $W \in \mathcal{W}$. However, in this section, we are only interested in the Lorentz group.

Lemma 3.5.2.10: *The closure of $\mathcal{J}_0$ with respect to the strong topology will be denoted by $\overline{\mathcal{J}_0}$. We can define a strongly continuous homomorphism $\varphi : \overline{\mathcal{J}_0} \mapsto \mathcal{L}_+^\uparrow$ by*

$$U\mathcal{R}(W)U^* = \mathcal{R}(\varphi(U)W) \quad \forall W \in \mathcal{W}. \tag{3.5.2.2}$$

Furthermore, $\overline{\mathcal{J}_0}$ is a central extension of $\mathcal{L}_+^\uparrow$.

Proof: First we show that we can extend φ_0 by (3.5.2.2) to a homomorphism $\varphi : \overline{\mathcal{J}_0} \mapsto \mathcal{L}_+^\uparrow$. For an arbitrary but fixed $V \in \overline{\mathcal{J}_0}$ we consider a sequence $\{V_n\}$ in $\mathcal{J}_0$ strongly converging to V. It is sufficient to show that there is a suitable $L \in \mathcal{L}_+^\uparrow$ such that V satisfies (3.5.2.2) if we set $\varphi(V) = L$. Apparently, we can replace $\{V_n\}$ by any subsequence. Set $L_n = \varphi_0(V_n)$. There are suitable $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, $\{\gamma_n'\}$, and $\{t_n\}$ in $\mathbf{R}$ such that

$$L_n = R_3(\alpha_n)R_1(\beta_n)R_3(\gamma_n)v_1(t_n)R_3(\alpha_n')R_1(\beta_n')R_3(\gamma_n'),$$

where v_1 is the group of boosts in the direction of the x_1-axis and R_1, resp. R_3, is the group of rotations about the x_1-axis, resp. the x_3-axis. By considering a subsequence we can assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, $\{\gamma_n'\}$ converge. It is sufficient to show that the strong limit of

$$\{((J_{R_3(\alpha_n/2)W_R}J_{W_R})(J_{R_1(\beta_n/2)W_2^{(0)}}J_{W_2^{(0)}})(J_{R_3(\gamma_n/2)W_R}J_{W_R}))^*V_n$$
$$((J_{R_3(\alpha_n'/2)W_R}J_{W_R})(J_{R_1(\beta_n'/2)W_2^{(0)}}J_{W_2^{(0)}})(J_{R_3(\gamma_n'/2)W_R}J_{W_R}))^*\},$$

which is, according to Lemma 3.5.2.9, equal to $U_1 V U_2$, for suitable $U_1, U_2 \in \mathcal{J}_0$, acts as an element of $\mathcal{L}_+^\uparrow$ by (3.5.2.2). Note that the elements of this sequence act as $v_1(t_n)$. Therefore, we can assume $\alpha_n = \beta_n = \gamma_n = \alpha_n' = \beta_n' = \gamma_n' = 0$. The strongly convergent sequence $(U(a)$ corresponds to the group in Lemma 3.5.2.8)

$$V_n^{\pm 1} U(1) V_n^{\mp 1} \overset{(3.5.2.1)}{=} U(e^{\mp 2\pi t_n})$$

entails that there is not a subsequence $\{t_{k_n}\}$ of $\{t_n\}$ such that $t_{k_n} \to \infty$ or $t_{k_n} \to -\infty$ for the limit of $\{V_n^{\pm 1} U(1) V_n^{\mp 1}\}$ is not the identity. Hence, we can assume that $\{t_n\}$ converges to t_0 (replace $\{V_n\}$ by a suitable subsequence). The limit of $\{(J_{v_1(t_n/2)W_2^{(0)}} J_{W_2^{(0)}})^* V_n\}$ acts trivially on the net, i.e., its limit $(J_{v_1(t_0/2)W_2^{(0)}} J_{W_2^{(0)}})^* V$ acts also trivially on the net, since von Neumann algebras are closed with respect to the strong topology. V satisfies (3.5.2.2) if we set $\varphi(V) = v_1(t_0)$.

It remains to show that φ is continuous. The proof above implies that to an arbitrary sequence $\{V_n\}$ in $\overline{\mathcal{J}_0}$ which strongly converges to 1 there is not a subsequence of $\{\varphi(V_n)\}$ without an accumulation point. If a subsequence $\{\varphi(V_{k_n})\}$ would converge to a nontrivial element of $\mathcal{L}_+^\uparrow$, then the continuity of $W \mapsto J_W$ would imply

$$J_{W_1} = \lim_{n\to\infty} V_{k_n} J_{W_1} V_{k_n}^* = J_{\lim_n \varphi(V_{k_n}) W_1} = J_{W_2},$$

for suitable wedges $W_1 \neq \pm W_2$ in $\mathcal{W}_0$. We would obtain the contradiction $1 = \varphi(J_{W_1} J_{W_2}) = g_{W_1} g_{W_2} \neq 1$.

The continuity of φ could be proven without the additional assumption (v) which we used in this section. It already follows from (3.4.1) and (i) of the strong CGMA if we use the strong continuity of the translation group rather than of the group in Lemma 3.5.2.8. However, the proof above can also be applied to de Sitter space (where we do not have any translations).

Lemma 3.5.2.11: *φ is open and continuous.*

Proof: According to Lemma 3.5.2.10 it remains to prove the openness of φ. Hence, it suffices to show that φ maps an arbitrary neighborhood $\mathcal{V}$ of the identity in $\overline{\mathcal{J}_0}$ onto a neighborhood of the identity in $\mathcal{L}_+^\uparrow$. If we assume the contrary, then we can find $L_n \in \mathcal{L}_+^\uparrow \backslash \varphi(\mathcal{V})$ converging to 1. We can assume

$$L_n \;=\; R_3(\alpha_n)R_1(\beta_n)R_3(\gamma_n)v_1(t_n)R_3(\alpha_n')R_1(\beta_n')R_3(\gamma_n')$$

such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{t_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, and $\{\gamma_n'\}$ converge to 0. Set

$$\begin{aligned} A_n \;=\; & (J_{R_3(\alpha_n/2)W_R}J_{W_R})(J_{R_1(\beta_n/2)W_2^{(0)}}J_{W_2^{(0)}})(J_{R_3(\gamma_n/2)W_R}J_{W_R}) \\ & (J_{v_1(t_n/2)W_2^{(0)}}J_{W_2^{(0)}})(J_{R_3(\alpha_n'/2)W_R}J_{W_R})(J_{R_1(\beta_n'/2)W_2^{(0)}}J_{W_2^{(0)}})(J_{R_3(\gamma_n'/2)W_R}J_{W_R}). \end{aligned}$$

$\varphi(A_n) = L_n$ implies $A_n \notin \mathcal{V}$ in contradiction to $A_n \to 1$.

Theorem 3.5.2.12: *There is a representation $\phi : \mathrm{Sl}(2,\mathbf{C}) \mapsto \mathcal{J}_0$ which is strongly continuous and such that*

$$U\mathcal{R}(W)U^* \;=\; \mathcal{R}(\phi(U)W) \quad \forall W \in \mathcal{W},\ U \in \mathrm{Sl}(2,\mathbf{C}). \qquad (3.5.2.3)$$

Proof: We apply Moore's cohomology theory [34]. Note that $(\mathrm{Sl}(2,\mathbf{C}),\pi)$ is the universal covering group of the Lie group $\mathcal{L}_+^\uparrow$, where π is the canonical epimorphism (for semi-simple Lie groups, the topologically defined universal covering group agrees with the universal covering group in Moore's sense). $(\overline{\mathcal{J}_0},\varphi)$ is a central extension of $\mathcal{L}_+^\uparrow$ by the Polish group $\ker\varphi$ (recall Lemma 3.5.2.11 and that we assume that the Hilbert space is separable). Hence, we can conclude that there is a continuous homomorphism $\phi : \mathrm{Sl}(2,\mathbf{C}) \mapsto \overline{\mathcal{J}_0}$ such that $\varphi \circ \phi = \pi$. However, $\overline{\mathcal{J}_0}' = \mathcal{J}_0'$ (every element of $\overline{\mathcal{J}_0}$ is, up to a central element, contained in $\mathcal{J}_0$) implies

$$\phi(\mathrm{Sl}(2,\mathbf{C})) = \phi(\mathrm{Sl}(2,\mathbf{C}))' \subset \overline{\mathcal{J}_0}' = \mathcal{J}_0' \subset \mathcal{J}_0.$$

3.5.3 The Poincaré Group

Now we will prove that (3.4.1) and (v), resp. (v'), imply the existence of a strongly continuous representation of the covering group $\widetilde{\mathcal{P}}_+$ of $\mathcal{P}_+$ which is generated by the universal covering group $\widetilde{\mathcal{P}}_+^\uparrow$ of $\mathcal{P}_+^\uparrow$ (resp. $\mathrm{SL}(2,\mathrm{C})$ and the translation group) and an inversion I which commutes with the elements of $\mathrm{SL}(2,\mathrm{C})$, inverts translations, and whose square is equal to the nontrivial central element Z of $\widetilde{\mathcal{P}}_+^\uparrow$.

Lemma 3.5.3.1: *$\mathcal{J}_e$ is perfect.*

Proof: To arbitrary $W_1, W_2 \in \mathcal{W}$, there is a $P \in \mathcal{P}_+^\uparrow$ such that $PW_1 = W_2$. Let $K \in \mathcal{J}_e'$ act as P. Hence

$$\begin{aligned} J_{W_1}J_{W_2} &= J_{W_1}J_{PW_1} \\ &= J_{W_1}(KJ_{W_1}K^{-1}) \\ &= (J_{W_1}KJ_{W_1})K^{-1} \\ &\in \mathcal{J}_e', \end{aligned}$$

where we used $J_{W_1}\mathcal{J}_e'J_{W_1} = (J_{W_1}\mathcal{J}_eJ_{W_1})' = \mathcal{J}_e'$.

Theorem 3.5.3.2: *Assume (3.4.1), that $W \mapsto \mathcal{R}(W)$ is an order-preserving bijection, and (v) (note that (3.4.1) can be derived from the strong CGMA). There is a strongly continuous representation U of the covering group $\widetilde{\mathcal{P}}_+$ of $\mathcal{P}_+$ such* *that* $U(\widetilde{\mathcal{P}}_+) = \mathcal{J}$. *The adjoint action of elements of* $U(\widetilde{\mathcal{P}}_+^\uparrow) = \mathcal{J}_e$ *leaves the net* $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ *invariant and acts geometrically correctly.* $U(Z) = J_W J_{W'}$ *(*$W \in \mathcal{W}$ *arbitrary) leaves the net pointwise fixed. The inversion* $I \in \widetilde{\mathcal{P}}_+$ *satisfies*

$$U(I)\mathcal{R}(W)U(I)^* = \mathcal{R}(-W')' \quad \forall W \in \mathcal{W}.$$

Proof: Theorem 3.5.1.3, Lemma 3.5.1.4, and Theorem 3.5.2.12 imply that there is a strongly continuous representation $U : \widetilde{\mathcal{P}}_+^\uparrow \mapsto \mathcal{J}_e$. We want to show $U(\widetilde{\mathcal{P}}_+^\uparrow) = \mathcal{J}_e$. Define an epimorphism by

$$\overline{U} : \widetilde{\mathcal{P}}_+^\uparrow \times \mathcal{Z} \mapsto \mathcal{J}_e, \quad \overline{U}((g,h)) = U(g)h$$

($\mathcal{Z}$ denotes the center of $\mathcal{J}_e$). We can conclude

$$U(\widetilde{\mathcal{P}}_+^\uparrow) = \overline{U}(\widetilde{\mathcal{P}}_+^\uparrow \times 1) = \overline{U}(\widetilde{\mathcal{P}}_+^\uparrow \times \mathcal{Z})' = \mathcal{J}_e' = \mathcal{J}_e.$$

Let $\widetilde{R}_3(\varphi)$ be the lifting of the rotation group about the x_3-axis (in $\widetilde{\mathcal{P}}_+$), i.e., in particular, $Z = \widetilde{R}_3(2\pi)$. $J_{R_3(\pi/2)W_R} J_{W_R} U(\widetilde{R}_3(\pi))^{-1}$ leaves every wedge algebra fixed, i.e., it lies in the center of $\mathcal{J}_e$. We can conclude either

$$J_{R_3(\pi/2)W_R} J_{W_R} = U(\widetilde{R}_3(\pi)) \text{ or } J_{R_3(\pi/2)W_R} J_{W_R} = U(\widetilde{R}_3(\pi))U(Z) \qquad (3.5.3.1)$$

and then, in any case,

$$\begin{aligned} J_{R_3(\pi/2)W_R} J_{W_R} &= U(\widetilde{R}_3(\pi/2)) J_{R_3(\pi/2)W_R} J_{W_R} U(\widetilde{R}_3(-\pi/2)) \\ &= J_{R_3(\pi)W_R} J_{R_3(\pi/2)W_R}. \end{aligned}$$

This proves (use (3.5.3.1) and $Z^2 = 1$)

$$\begin{aligned} U(Z) &= U(\widetilde{R}_3(\pi))^2 \\ &= (J_{R_3(\pi/2)W_R} J_{W_R})^2 \end{aligned}$$

$$\begin{aligned} &= (J_{R_3(\pi)W_R} J_{R_3(\pi/2)W_R})(J_{R_3(\pi/2)W_R} J_{W_R}) \\ &= J_{R_3(\pi)W_R} J_{W_R} \\ &= J_{W_R'} J_{W_R}. \end{aligned}$$

According to (3.4.1), $J_{W_R'} J_{W_R}$ leaves the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ pointwise fixed, i.e., it commutes with any modular conjugation. To every $W \in \mathcal{W}$ there is a $K \in \mathcal{J}_e$ which maps W_R onto $W \in \mathcal{W}$, i.e., such that

$$J_{W_R'} J_{W_R} = K J_{W_R'} J_{W_R} K^{-1} = J_{W'} J_W.$$

We still have to prove that U can be extended to a representation of $\widetilde{\mathcal{P}}_+$. Set $U(I) = J_{W_3^{(0)}} U(\widetilde{R}_3(\pi))$. Note

$$\begin{aligned} U(I)^2 &= J_{W_3^{(0)}} U(\widetilde{R}_3(\pi)) J_{W_3^{(0)}} U(\widetilde{R}_3(\pi)) \\ &= J_{W_3^{(0)}} J_{W_3^{(0)}} U(\widetilde{R}_3(\pi)) U(\widetilde{R}_3(\pi)) \\ &= U(\widetilde{R}_3(2\pi)) \\ &= U(Z). \end{aligned}$$

We conclude $(U(I) J_W \in \mathcal{J}_e)$

$$\begin{aligned} U(I)\mathcal{R}(W)U(I)^* &= J_{W_3^{(0)}} U(\widetilde{R}_3(\pi)) \mathcal{R}(W) (J_{W_3^{(0)}} U(\widetilde{R}_3(\pi)))^* \\ &= J_{W_3^{(0)}} U(\widetilde{R}_3(\pi)) J_W \mathcal{R}(W)' J_W U(\widetilde{R}_3(-\pi)) J_{W_3^{(0)}} \\ &= \mathcal{R}(-g_W W)' \\ &= \mathcal{R}(-W')' \end{aligned}$$

$\forall W \in \mathcal{W}$. In particular, we obtain

$$U(I)\mathcal{R}(W)U(I)^* = \mathcal{R}(W)'$$

if $W \in \mathcal{W}_0$ ($\mathcal{W}_0$ denotes the set of wedges which contain the origin in their edge), i.e.,

$$U(I)J_W U(I)^* = J_W \quad \forall W \in \mathcal{W}_0.$$

Hence, $U(I)$ commutes with $U(\mathrm{SL}(2,\mathbf{C}))$. $U(I)V(x)U(I)^* = V(-x)$ follows from Lemma 3.5.1.4. Therefore, U can be extended to a representation of $\widetilde{\mathcal{P}}_+$, which is strongly continuous. $U(\widetilde{\mathcal{P}}_+^\uparrow) = \mathcal{J}_e$ implies $U(\widetilde{\mathcal{P}}_+) = \mathcal{J}$.

Corollary 3.5.3.3: *If locality is fulfilled, in addition to the assumptions in Theorem 3.5.3.2, we obtain a representation U of $\mathcal{P}_+$. The representation satisfies*

$$U(g_W) = J_W \quad \forall W \in \mathcal{W}.$$

It acts geometrically correctly and $U(\mathcal{P}_+) = \mathcal{J}$. Furthermore, the adjoint action of modular conjugations leaves $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ invariant and the action is given by

$$J_{W_0}\mathcal{R}(W)J_{W_0} = \mathcal{R}(g_{W_0}W) \quad \forall W, W_0 \in \mathcal{W}.$$

In particular, Haag duality is fulfilled.

Proof: We obtain a representation of $\mathcal{P}_+$ if and only if $J_W J_{W'} = U(Z) = 1$ for $W \in \mathcal{W}$. Assume locality. We can conclude

$$\begin{aligned}
\mathcal{R}(W) &= \mathcal{R}(g_{W'}g_W W) \\
&= J_{W'}J_W\mathcal{R}(W)J_W J_{W'} \\
&= J_{W'}\mathcal{R}(W)'J_{W'} \\
&\supset J_{W'}\mathcal{R}(W')J_{W'} \\
&= \mathcal{R}(W')' \\
&\supset \mathcal{R}((W')') \\
&= \mathcal{R}(W),
\end{aligned}$$

i.e., $\mathcal{R}(W)' = \mathcal{R}(W')\ \forall W \in \mathcal{W}$. Haag duality is satisfied and $U(Z) = 1$. The adjoint action of J_{W_0} leaves $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant:

$$\begin{aligned} J_{W_0}\mathcal{R}(W)J_{W_0} &= J_{W_0}J_W\mathcal{R}(W)'J_WJ_{W_0} \\ &= \mathcal{R}(g_{W_0}g_WW') \\ &= \mathcal{R}(g_{W_0}W) \end{aligned}$$

$\forall W \in \mathcal{W}$. $U(\mathcal{P}_+) = \mathcal{J}$ implies that there is exactly one element in $\mathcal{J}$ which acts as g_W for $W \in \mathcal{W}$, i.e., $U(g_W) = J_W$.

Theorem 3.5.3.4: *The conclusions of Theorem 3.5.3.2 and Corollary 3.5.3.3 still hold if we assume (v') instead of (v).*

Proof: We can proceed as above if we replace in the proof of 3.5.2.7 $\mathcal{R}_a \cap \mathcal{R}_b$ by $\cap_{W\supset V(a)W_R\cap V(b)W_R}\mathcal{R}(W)$. Note

$$\mathcal{R}(V(c)W_R) \supset \cap_{W\supset V(a)W_R\cap V(b)W_R}\mathcal{R}(W),$$

for $a \leq c \leq b$.

3.6 Geometric Action of Modular Groups

It will be shown that modular covariance is fulfilled if the adjoint action of modular groups of wedge algebras leaves the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant and if the corresponding geometric action of the group elements can be implemented by point transformations on Minkowski space. To ensure the latter condition we proceed as above, i.e., we assume that the strong CGMA is satisfied with (iv) replaced by:

(iv') The adjoint action of elements of the modular groups Δ^{it}, $t \in \mathbf{R}$, of pairs $(\mathcal{R}(W_0), \Omega)$, $W_0 \in \mathcal{W}$, leaves the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ invariant, i.e., for $W \in \mathcal{W}$,

$$\Delta^{it}\mathcal{R}(W)\Delta^{-it} = \mathcal{R}(W_t),$$

$\forall t \in \mathbf{R}$ and suitable wedges $W_t \in \mathcal{W}$.

Our assumptions ensure that the geometric action of the modular groups Δ^{it}, $t \in \mathbf{R}$, of wedge algebras is implemented by elements L_t of the extended Poincaré group, i.e.,

$$\Delta^{it}\mathcal{R}(W)\Delta^{-it} = \mathcal{R}(L_t W) \ \forall t \in \mathbf{R}, \ \forall W \in \mathcal{W} \tag{3.6.1}$$

(see Theorem 3.1.3). It will be shown that under our assumptions the modular groups act geometrically as we expect from [4, 5], i.e., that modular covariance is fulfilled. In the following this means that if Δ^{it}, $t \in \mathbf{R}$, is the modular group of the wedge algebra $\mathcal{R}(W_R)$, then we find

$$L_t x = \begin{pmatrix} \cosh(2\pi t) & \sinh(\pm 2\pi t) & 0 & 0 \\ \sinh(\pm 2\pi t) & \cosh(2\pi t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x \quad \forall x \in \mathbf{R}^4, \ t \in \mathbf{R} \tag{3.6.2}$$

(- for a representation of positive energy and + for a representation of negative energy). The geometric action of modular groups of other wedge algebras can be derived from (3.6.2).

In this section we will assume that the Hilbert space is separable. (3.6.2) implies that the modular groups induce a representation of the covering group of the Poincaré group [13].

Modular covariance implies that the group generated by the modular groups is identical to the group consisting of even products of modular conjugations. As shown in [25] (the proofs are also applicable to our slightly different setting), the adjoint action of modular conjugations leaves the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ fixed if and only if locality is fulfilled. In this case, we even obtain a representation of the Poincaré group $\mathcal{P}_+^\uparrow$ (see also [8]). Furthermore, we find the same geometric action of the modular objects as in (3.3.1), resp. [4, 5].

3.6.1 Modular Groups

Theorem 3.6.1.1: *If the strong CGMA is satisfied (with (iv) replaced by (iv')), then the modular groups of the wedge algebras generate a representation of the covering group of the Poincaré group $\mathcal{P}_+^\uparrow$ of positive or negative energy. Furthermore, the groups act geometrically as the associated Lorentz boosts (see (3.6.2)), i.e., modular covariance is fulfilled.*

Proof: Call the group generated by the modular groups $\mathcal{M}$ and the corresponding subgroup of the extended Poincaré group $\mathcal{K}$. Condition (i) of the strong CGMA and Theorem 3.1.3 ensure that there is indeed a well-defined homomorphism $\pi_g : \mathcal{M} \to \mathcal{K}$ defined by

$$G\mathcal{R}(W)G^* = \mathcal{R}(\pi_g(G)W), \quad \text{for} \quad G \in \mathcal{M},\ W \in \mathcal{W},$$

since only the identity of the extended Poincaré group acts trivially on the set of wedge algebras. Repeatedly, we will use that every $G \in \mathcal{M}$ with $\pi_g(G) = 1$ lies in the center of $\mathcal{M}$ because of the uniqueness of the modular groups ($G\mathcal{R}(W)G^* = \mathcal{R}(W)$ for every $W \in \mathcal{W}$).

Set $\mathcal{A}_c = \mathcal{R}(W_R + c)$, for $c \in \mathbf{R}^4$. Let Δ_c^{it}, $t \in \mathbf{R}$, be the modular group of $(\mathcal{A}_c, \Omega)$. We will determine the geometric action of the modular group Δ_0^{it}, $t \in \mathbf{R}$. By using Theorem 3.1.3 we obtain $\forall t \in \mathbf{R}$ an element $L_t = \pi_g(\Delta_0^{it})$ of the extended Poincaré group. To every L_t there exists an $a_t \in \mathbf{R}^4$, a real number $k_t \neq 0$ and a homogeneous Lorentz transformation Λ_t such that $L_t x = k_t \Lambda_t x + a_t \ \forall x \in \mathbf{R}^4$. $L_t = (L_{\frac{t}{2}})^2$ implies $k_t > 0$ and $\Lambda_t \in \mathcal{P}_+^\uparrow$. The geometric action of Δ_0^{it} can be described by

$$\Delta_0^{it} \mathcal{R}(W[e_1, e_2, a]) \Delta_0^{-it} = \mathcal{R}(W[\Lambda_t e_1, \Lambda_t e_2, L_t a]),$$

for arbitrary $a \in \mathrm{R}^4$ and linearly independent lightlike vectors e_1, e_2 in the forward lightcone.

$\Lambda_t l_{1+}$ is a multiple of l_{1+} and $\Lambda_t l_{1-}$ is a multiple of l_{1-} because of $\Lambda_t \in \mathcal{P}_+^\uparrow$ and $\Delta_0^{it} \mathcal{A}_0 \Delta_0^{-it} = \mathcal{A}_0$.

Therefore, L_t has to be of the following form (with $e^{\beta_t} = k_t$, i.e., $e^{-\beta_t} L_t \in \mathcal{P}_+^\uparrow$)

$$L_t x = e^{\beta_t} \begin{pmatrix} \cosh(\alpha_t) & \sinh(\alpha_t) & \gamma_{1t} & \gamma_{2t} \\ \sinh(\alpha_t) & \cosh(\alpha_t) & \gamma_{3t} & \gamma_{4t} \\ 0 & 0 & \gamma_{5t} & \gamma_{6t} \\ 0 & 0 & \gamma_{7t} & \gamma_{8t} \end{pmatrix} x + \begin{pmatrix} \delta_{1t} \\ \delta_{2t} \\ \delta_{3t} \\ \delta_{4t} \end{pmatrix}, \quad \forall x \in \mathbf{R}^4,$$

for suitable $\alpha_t, \beta_t, \gamma_{1t}, \gamma_{2t}, \gamma_{3t}, \gamma_{4t}, \gamma_{5t}, \gamma_{6t}, \gamma_{7t}, \gamma_{8t}, \delta_{1t}, \delta_{2t}, \delta_{3t}, \delta_{4t} \in \mathbf{R}$. By using $\mathcal{A}_0 = \Delta_0^{it} \mathcal{A}_0 \Delta_0^{-it}$ we conclude $\delta_{1t} = \delta_{2t} = 0$.

α_t and β_t are uniquely defined for

$$e^{\beta_t} \begin{pmatrix} \cosh(\alpha_t) & \sinh(\alpha_t) \\ \sinh(\alpha_t) & \cosh(\alpha_t) \end{pmatrix} = e^{\beta'_t} \begin{pmatrix} \cosh(\alpha'_t) & \sinh(\alpha'_t) \\ \sinh(\alpha'_t) & \cosh(\alpha'_t) \end{pmatrix}$$

implies

$$e^{\beta_t - \beta'_t} = \begin{pmatrix} \cosh(\alpha'_t - \alpha_t) & \sinh(\alpha'_t - \alpha_t) \\ \sinh(\alpha'_t - \alpha_t) & \cosh(\alpha'_t - \alpha_t) \end{pmatrix},$$

respectively $\alpha_t = \alpha'_t$ and $\beta_t = \beta'_t$.

$t \mapsto \alpha_t$ and $t \mapsto \beta_t$ are group homomorphisms of $(\mathbf{R}, +)$ into $(\mathbf{R}, +)$ for π_g is a homomorphism. Now we want to show that the homomorphisms $t \mapsto \pm\alpha_t + \beta_t$ are continuous. If $t \mapsto \alpha_t + \beta_t$ is not continuous, then there are $t_n \in \mathbf{R}$ such that $t_n \to 0$ and either

$$\alpha_{t_n} + \beta_{t_n} > c > 0 \quad \forall n \in \mathbf{N} \quad \text{or} \quad \alpha_{t_n} + \beta_{t_n} < c < 0 \quad \forall n \in \mathbf{N},$$

for a suitable c. It is enough to consider the first case (replace t_n by $-t_n$ in the other case). $\Delta_0^{it_n}\mathcal{A}_{l_{1+}}\Delta_0^{-it_n} \subset \mathcal{A}_{e^c l_{1+}}$ follows from $L_t l_{1+} \equiv e^{\alpha_t+\beta_t} l_{1+}$ modulo vectors in the direction of the x_2-axis and the x_3-axis. Choose $A \in \mathcal{A}_{l_{1+}}$ and $B \in \mathcal{A}'_{e^c l_{1+}}$ which do not commute. Then

$$[\Delta_0^{it_n} A \Delta_0^{-it_n}, B] \;=\; 0,$$

however, $\{[\Delta_0^{it_n} A \Delta_0^{-it_n}, B]\}$ converges weakly against $[A, B] \neq 0$. This proves the continuity of $t \mapsto \alpha_t + \beta_t$. A similar argument shows that $t \mapsto -\alpha_t + \beta_t$ is continuous. It follows $\alpha_t = \lambda t$ and $\beta_t = \mu t$, for suitable $\lambda, \mu \in \mathbf{R}$, which are independent of t [27, I.2.12.].

$$(\Lambda_t(0,1,0,0)^{\mathrm{T}}, \Lambda_t(0,0,1,0)^{\mathrm{T}}) \;=\; (\Lambda_t(1,0,0,0)^{\mathrm{T}}, \Lambda_t(0,0,1,0)^{\mathrm{T}}) \;= 0$$

implies

$$\sinh(\lambda t)\gamma_{1t} = \cosh(\lambda t)\gamma_{3t} \quad \text{and} \quad \cosh(\lambda t)\gamma_{1t} = \sinh(\lambda t)\gamma_{3t},$$

i.e., $\gamma_{1t} = \gamma_{3t} = 0$. Similarly, it follows $\gamma_{2t} = \gamma_{4t} = 0$. $\begin{pmatrix} \gamma_{5t} & \gamma_{6t} \\ \gamma_{7t} & \gamma_{8t} \end{pmatrix}$ has to be a rotation about the x_1-axis ($\det \Lambda_t = \det \Lambda_{\frac{t}{2}} \Lambda_{\frac{t}{2}}$). We obtain for L_t the form

$$L_t x \;=\; e^{\mu t} \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) & 0 & 0 \\ \sinh(\lambda t) & \cosh(\lambda t) & 0 & 0 \\ 0 & 0 & \cos\rho_t & \sin\rho_t \\ 0 & 0 & -\sin\rho_t & \cos\rho_t \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \delta_{3t} \\ \delta_{4t} \end{pmatrix} \qquad \forall x \in \mathbf{R}^4$$

with suitable $\rho_t \in \mathbf{R}$. If we call the elements in $\mathcal{M}$ associated to the modular group of $(\mathcal{A}_{l_{1+}}, \Omega)$ L'_t, i.e., $L'_t = \pi_g(\Delta_{l_{1+}}^{it})$, then, $\forall x \in \mathbf{R}^4$,

$$L'_t x \;=\; e^{\mu' t} \begin{pmatrix} \cosh(\lambda' t) & \sinh(\lambda' t) & 0 & 0 \\ \sinh(\lambda' t) & \cosh(\lambda' t) & 0 & 0 \\ 0 & 0 & \cos\rho'_t & \sin\rho'_t \\ 0 & 0 & -\sin\rho'_t & \cos\rho'_t \end{pmatrix} x + \begin{pmatrix} \nu'_{1t} \\ \nu'_{2t} \\ \delta'_{1t} \\ \delta'_{2t} \end{pmatrix},$$

for suitable $\lambda', \mu', \rho'_t, \delta'_{1t}, \delta'_{2t} \in \mathbf{R}$ and with $\nu'_{1t} = \nu'_{2t} = 1 - e^{(\lambda'+\mu')t}$ (proceed as above and use $\Delta_{l_{1+}}^{it}\mathcal{A}_{l_{1+}}\Delta_{l_{1+}}^{-it} = \mathcal{A}_{l_{1+}}$ implying $L'_t l_{1+} \equiv l_{1+}$ modulo vectors in the direction of the x_2-axis and the x_3-axis).

$(\mathcal{A}_{l_{1+}} \subset \mathcal{A}_0, \Omega)$ is a half-sided modular inclusion. This follows easily from the form of L_t. We can define a group $W_1(s)$, $s \in \mathbf{R}$, by $W_1(\mp(e^{\pm 2\pi t} - 1)) = L_t L'_{-t}$ [40, 42]. We know

$$L_t W_1(1) L_{-t} = W_1(e^{\mp 2\pi t}) \tag{3.6.1.1}$$

and

$$W_1(\pm 1)\, W[l_{1+}, l_{1-}, 0] = W[l_{1+}, l_{1-}, l_{1+}]. \tag{3.6.1.2}$$

L_t and L'_s commute modulo translations, i.e., $L_t W_1(1) L_{-t} = W_1(e^{\mp 2\pi t})$ does not depend on t (modulo translations). This implies that W_1 is a group of translations. It follows $\lambda = \lambda'$, $\mu = \mu'$, $\rho_t = \rho'_t$, and

$$W_1(s)x = x \pm \begin{pmatrix} s \\ s \\ k_s \\ l_s \end{pmatrix} \quad \forall x \in \mathbf{R}^4, \quad \forall s \in \mathbf{R}, \tag{3.6.1.3}$$

for suitable $k_s, l_s \in \mathbf{R}$. Furthermore, (3.6.1.1) and (3.6.1.3) entail

$$\lambda + \mu \in \{-2\pi, 2\pi\}. \tag{3.6.1.4}$$

The half-sided modular inclusion $(\mathcal{A}_{-l_{1-}} \subset \mathcal{A}_0, \Omega)$ provides a second group $W_2(t)$, $t \in \mathbf{R}$, of translations which lies in $\mathcal{M}$. (3.6.1.3) and the corresponding identitiy for $W_2(1)$ entail

$$W_1(\pm 1)x = x + \begin{pmatrix} 1 \\ 1 \\ g_1 \\ h_1 \end{pmatrix} \quad \text{and} \quad W_2(\pm 1)x = x + \begin{pmatrix} -1 \\ 1 \\ g_2 \\ h_2 \end{pmatrix} \quad \forall x \in \mathbf{R}^4.$$

$W_1(\pm 1) W_2(\pm 1)$ is a translation by $(0, 2, g_1 + g_2, h_1 + h_2)^{\mathrm{T}}$, for a suitable choice of the signs. By choosing a suitable coordinate system (rotate about the x_1-axis if necessary), we can assume $g_1 + g_2 = 0$. To every element of the modular group of

$(\mathcal{R}(W[l_{2+}, l_{2-}, 0]), \Omega)$ there corresponds an element K_t in $\mathcal{M}$. We can prove a general form of K_t similarly as for L_t. In $\mathbf{R}^2$, a nontrivial composition of a dilatation and a rotation does not commute with nontrivial translations. Therefore, the existence of the nontrivial translation $W_1(\pm 1)W_2(\pm 1)$, which lies in $\mathcal{M}$, leaves the wedge $W[l_{2+}, l_{2-}, 0]$ fixed and necessarily commutes with K_t (a unitary operator leaving the vacuum fixed and acting geometrically as this translation commutes with the modular group of $(\mathcal{R}(W[l_{2+}, l_{2-}, 0]), \Omega)$ because of the uniqueness of the modular group) implies that K_t is of the form

$$K_t x = \begin{pmatrix} \cosh(\omega t) & 0 & \sinh(\omega t) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh(\omega t) & 0 & \cosh(\omega t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ d'_1 \\ 0 \\ d'_2 \end{pmatrix} t, \quad \forall x \in \mathbf{R}^4, \tag{3.6.1.5}$$

for suitable $d'_1, d'_2 \in \mathbf{R}$, $\omega = \pm 2\pi$. By using the half-sided modular inclusions $(\mathcal{R}(W[l_{2+}, l_{2-}, \pm l_{2\pm}]) \subset \mathcal{R}(W[l_{2+}, l_{2-}, 0]), \Omega)$ we obtain again two groups of translations (compare $W_1(s)$, $W_2(t)$). (3.6.1.5) and the property corresponding to (3.6.1.1) entail that the parts of the translations in the direction of the x_1-axis and the x_3-axis are trivial, i.e., $\mathcal{M}$ contains the translations in the directions of l_{2+} and l_{2-} (compare (3.6.1.3)) and, therefore, nontrivial translations in the direction of the x_2-axis. This implies $\mu = \rho_t = 0$ if we use the uniqueness of the modular group of $(\mathcal{A}_0, \Omega)$ (such translations commute with L_t). Thus, we have proven that L_t is of the form

$$L_t x = \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) & 0 & 0 \\ \sinh(\lambda t) & \cosh(\lambda t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ \delta_{3t} \\ \delta_{4t} \end{pmatrix} \quad \forall x \in \mathbf{R}^4,$$

for suitable $\lambda, \delta_{3t}, \delta_{4t} \in \mathbf{R}$. Furthermore, we know $\lambda = \pm 2\pi$ (see (3.6.1.4)). For arbitrary wedge algebras we can derive a similar form for the geometric action of the modular groups.

By using a finite number of modular groups of wedge algebras we can generate $\mathcal{M}$, which is a central extension of the Poincaré group (see above), in the following way. Choose modular groups such that we obtain a nontrivial translation group. For example, the modular inclusion $(\mathcal{A}_{l_{1+}} \subset \mathcal{A}_0, \Omega)$ provides a nontrivial translation group in the direction of l_{1+}. Furthermore, we need three modular groups which act geometrically (modulo translations) as the boosts in the directions of the x_1-, x_2- and x_3-axes. We obtain a group which acts transitively on $\mathcal{W}$, i.e., it is identical to $\mathcal{M}$ and $\pi_g(\mathcal{M}) = \mathcal{K} = \mathcal{P}_+^\uparrow$. There is a unitary $A \in \mathcal{M}$ which acts geometrically by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then A commutes with Δ_0^{it} $\forall t \in \mathbf{R}$ $(A^* \mathcal{A}_0 A = \mathcal{A}_0)$, i.e., $\delta_{3t} = \delta_{4t} = 0$ $\forall t \in \mathbf{R}$. Using the transitivity of $\mathcal{M}$ on the set of wedges we can conclude that the modular inclusions $(\mathcal{R}(W[e_1, e_2, e_1]) \subset \mathcal{R}(W[e_1, e_2, 0]), \Omega)$ (e_1, e_2 arbitrary linearly independent lightlike vectors in the forward lightcone) are all either +half-sided modular inclusions or all -half-sided modular inclusions. As in [13], we can prove that $\mathcal{M}$ gives rise to a representation of the covering group of the Poincaré group of positive or of negative energy depending on the kind of half-sided modular inclusions we obtain above.

3.6.2 Modular Conjugations

Theorem 3.6.2.1: *Assume that the strong CGMA (with (iv) replaced by (iv')) is fulfilled. The group generated by the modular groups of wedge algebras is equal to the group consisting of the even products of modular conjugations. The adjoint action of the modular conjugations leaves the net of wedge algebras invariant if and only if locality is fulfilled, i.e., if and only if* $\mathcal{R}(W') \subset \mathcal{R}(W)'$ *for* $W \in \mathcal{W}$. *The modular conjugations act geometrically as in (3.3.1) in this case. Furthermore, we* *obtain a representation of the Poincaré group $\mathcal{P}_+^\uparrow$ and wedge duality is fulfilled.*

Proof: Let $\mathcal{M}$ be the group generated by the modular groups of wedge algebras and $\pi_g : \mathcal{M} \to \mathcal{P}_+^\uparrow$ the epimorphism induced by the geometric action. $\mathcal{M}$ is isomorphic to the Poincaré group $P_+^\uparrow$ or to the covering group of $P_+^\uparrow$. We know that the adjoint action of J_{W_R} leaves $\mathcal{M}$ invariant [13]. In addition,

$$\pi_g(J_{W_R} G J_{W_R}) = g_{W_R} \pi_g(G) g_{W_R} \quad \forall G \in \mathcal{M} \tag{3.6.2.1}$$

with

$$g_{W_R} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It will be shown that the product of two modular conjugations lies in $\mathcal{M}$. The adjoint action of elements of $\mathcal{M}$ maps wedge algebras onto wedge algebras, i.e., modular conjugations onto modular conjugations. If $\mathcal{R}(W_1)$, $\mathcal{R}(W_2)$ are two wedge algebras with modular conjugations J_1, J_2, then there exists a $G \in \mathcal{M}$ such that $G\mathcal{R}(W_1)G^* = \mathcal{R}(W_2)$, i.e.,

$$J_1 J_2 = J_1(G J_1 G^*) = (J_1 G J_1) G^* \in \mathcal{M}.$$

Therefore, the group consisting of the even products of modular conjugations, which contains all translations, is a normal subgroup of $\mathcal{M}$. It follows easily that this group is identical to $\mathcal{M}$.

$(\mathcal{R}(W_R)' = J_{W_R}\mathcal{R}(W_R)J_{W_R}, \Omega)$ has the same modular group as $(\mathcal{R}(W_R'), \Omega)$, i.e., if $\mathcal{R}(W_R)'$ is a wedge algebra, it must be equal to $\mathcal{R}(W_R')$, since the algebras are not abelian (an abelian von Neumann algebra with cyclic vector is maximally abelian) and since the elements of modular groups associated to wedges different from W and W' act geometrically differently. In this case, wedge duality is fulfilled.

Locality implies wedge duality for the adjoint action of the modular group of $(\mathcal{R}(W_R), \Omega)$, hence of $(J_{W_R}\mathcal{R}(W_R)J_{W_R}, \Omega)$, leaves $\mathcal{R}(W_R') \subset J_{W_R}\mathcal{R}(W_R)J_{W_R}$ invariant.

It remains to show that wedge duality implies the invariance of the net with respect to the adjoint action of modular conjugations J_W, $W \in \mathcal{W}$. To any $W_0 \in \mathcal{W}$, there is a $G \in \mathcal{M}$ such that $\mathcal{R}(W_0) = G\mathcal{R}(W)G^{-1}$, i.e.,

$$J_W\mathcal{R}(W_0)J_W \;=\; (J_W G J_W)J_W\mathcal{R}(W)J_W(J_W G J_W)^{-1},$$

which is a wedge algebra if wedge duality is satisfied (recall (3.6.2.1)).

CHAPTER 4
GEOMETRIC MODULAR ACTION ON DE SITTER SPACE

We can embed de Sitter space into five-dimensional Minkowski space

$$dS^4 \; = \; \{x \in \mathbf{R}^5 \,|\, x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 \; = \; -1\}$$

with the induced metric and causal structure from Minkowski space. Again we use wedgelike regions in order to define $\mathcal{W}$:

$$\mathcal{W} \; = \; \{W \cap dS^4 \,|\, W \text{ is a wedge in five-dimensional Minkowski space}\}$$

For future-directed lightlike and linearly independent k, l we define an element of $\mathcal{W}$ by

$$W[k,l] \; = \; \{\alpha k - \beta l + h \in dS^4 \,|\, \alpha, \beta > 0, (k,h) = (l,h) = 0\}.$$

Similarly, we can define $\mathcal{W}$ and wedges $W[k,l]$ for three-dimensional de Sitter space dS^3. We denote by $\mathcal{S}$ the de Sitter group, which can be identified with the Lorentz group of five-dimensional Minkowski space.

Motivated by Bisognano-Wichmann type theorems for wedge algebras [9, 12] we choose again the set of wedges $\mathcal{W}$ and consider a net of von Neumann algebras $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ acting on a Hilbert space $\mathcal{H}$ and satisfying the CGMA for a vector $\Omega \in \mathcal{H}$.

Since the third condition of our strong CGMA is trivially satisfied for arbitrary pairs of wedges in de Sitter space, we will use in this chapter the following modified form of the strong CGMA.

Strong Condition of Geometric Modular Action: Let $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ be a net of von Neumann algebras, which act on a Hilbert space $\mathcal{H}$. Let $\Omega \in \mathcal{H}$. $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}}, \Omega)$ satisfies the strong CGMA if the following conditions are satisfied.

(i) $W \mapsto \mathcal{R}(W)$ is a bijection.

(ii) Ω is cyclic and separating for $\mathcal{R}(W)$, $W \in \mathcal{W}$.

(iii) For $W_1, W_2, W \in \mathcal{W}$ with $W_1 \cap W_2 \neq \emptyset$, there holds

$$\mathcal{R}(W_1) \cap \mathcal{R}(W_2) \subset \mathcal{R}(W) \quad \Leftrightarrow \quad W_1 \cap W_2 \subset W.$$

(iv) The adjoint action of J_W, $W \in \mathcal{W}$, leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant.

We will show that the modular conjugations act as Lorentz transformations on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ if the strong CGMA is satisfied. Furthermore, the modular conjugations of the wedge algebras induce a strongly continuous representation of the proper de Sitter group if the modular conjugations act transitively on the net. In order to prove the latter statement we have to assume, in addition, that Ω is cyclic for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$ if $W_1, W_2 \in \mathcal{W}$, $W_1 \cap W_2 \neq \emptyset$.

In [9] vacuum states of quantum field theories on de Sitter space were studied. Under the assumption of a state satisfying the KMS condition for wedge algebras, it was shown that the wedge algebras possess the Reeh-Schlieder property, i.e., the modular objects of these algebras associated to the vacuum are defined. With this it can be concluded that the modular objects act geometrically, resp. that the CGMA is satisfied.

4.1 Bijections of Wedges

The modular conjugations act on $\mathcal{W}$ as bijections $\tau : \mathcal{W} \mapsto \mathcal{W}$ which satisfy, for $W_1, W_2, W \in \mathcal{W}$ with $W_1 \cap W_2 \neq \emptyset$ and $\tau W_1 \cap \tau W_2 \neq \emptyset$,

$$W_1 \cap W_2 \subset W \Leftrightarrow \tau W_1 \cap \tau W_2 \subset \tau W. \tag{4.1.1}$$

In the following we will consider a bijection $\tau : \mathcal{W} \mapsto \mathcal{W}$ fulfilling (4.1.1).

We will call two (unordered) pairs of wedges (W_1, W_2), (W_3, W_4) dual iff $\emptyset \neq W_1 \cap W_2 = W_3 \cap W_4$. A pair (W_1, W_2) is called self-dual if there is no other pair which is dual to (W_1, W_2). For $W_1 = W[l_1, l_2]$ and $W_2 = W[k_1, k_2]$ (l_1, l_2, k_2, k_2 suitable future-directed lightlike vectors), this is equivalent to $l_1 = k_1$ or $l_2 = k_2$ (up to multiples) [21].

We will establish (4.1.1) for arbitrary $W, W_1, W_2 \in \mathcal{W}$ (Lemma 4.1.9). Note that τ maps a self-dual pair of wedges (W_1, W_2) onto a self-dual pair $(\tau W_1, \tau W_2)$ if $\tau W_1 \cap \tau W_2 \neq \emptyset$. This fact will play an important role in the following.

Again we can characterize pairs of wedges with trivial intersection.

Lemma 4.1.1: *The wedges W_R and $W_1 = W[l_{2+}, (1, a, b, c, d)^{\mathrm{T}}]$ ($a, b, c, d \in \mathbf{R}$, $a^2 + b^2 + c^2 + d^2 = 1$) possess trivial intersection if and only if $0 < a \leq 1$, $0 \leq b < 1$, and $c = 0 = d$. The statement is still true if we replace W_1 by W_1' and the condition $0 < a \leq 1$ by $-1 \leq a < 0$ or also if we replace l_{2+} by l_{2-} and $0 \leq b < 1$ by $-1 < b \leq 0$.*

Proof: see the proof of Lemma 3.1.6

Lemma 4.1.2: *If $\{W_i \,|\, i \in I\}$ is an uncountable set of wedges, then there is an infinite subset $I_0 \subset I$ such that $\cap_{i \in I_0} W_i \neq \emptyset$.*

Proof: Let $\mathcal{B}$ be a countable basis of the topology of dS^4. For any W_i, we can choose an arbitrary but fixed $\emptyset \neq B_i \in \mathcal{B}$ such that $B_i \subset W_i$, i.e., there is a map $\varphi : I \mapsto \mathcal{B}$ such that $\varphi(i) = B_i \subset W_i$. I is uncountable but $\mathcal{B}$ is countable implying that there exists a $B \in \mathcal{B}$ which is the image of infinitely many $i \in I$. Set $I_0 = \varphi^{-1}(\{B\})$.

Lemma 4.1.3: *To any future-directed lightlike vector l there are future-directed lightlike l_1, l_2, l_3, l_4 such that any four of the vectors l, l_1, l_2, l_3, l_4 are linearly independent and*

$$\cap_{k=1}^{4} \tau W[l, l_k] \neq \emptyset.$$

Proof: Set $l_t = (\sqrt{1 + t^2 + t^4 + t^6}, 1, t, t^2, t^3)^{\mathrm{T}}$ for $t \in \mathbf{R}$. We can assume $l = l_0$. $l_{t_1}, l_{t_2}, l_{t_3}, l_{t_4}$ are linearly independent for pairwise different $t_1, t_2, t_3, t_4 \in \mathbf{R}$, since this implies

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 \\ t_1^3 & t_2^3 & t_3^3 & t_4^3 \end{vmatrix} \neq 0.$$

Apply Lemma 4.1.2 to $\{W[l_0, l_t] \,|\, t \in \mathbf{R} \backslash \{0\}\}$ (in three-dimensional de Sitter space the argument can still be used for $l_t = (2t^4 + 1, 2t^2 - 1, 2t, 2t^4)^{\mathrm{T}}$).

Lemma 4.1.4:

$$W_R \cap W[l_{1+}, (1, a_1, b_1, 0, 0)^{\mathrm{T}}] \;\subset\; W[l_{1+}, (1, a_2, b_2, 0, 0)^{\mathrm{T}}] \qquad (4.1.2)$$

for $a_1^2 + b_1^2 = 1 = a_2^2 + b_2^2$, $b_1 b_2 \geq 0$, $a_1 \neq 1 \neq a_2$, *and* $a_2 \leq a_1$. *Furthermore,* $W[l, l_1] \cap W[l, l_2] \subset W[l, l_3]$, *for future-directed lightlike* l, l_1, l_2, l_3, *is only possible if the wedges are, up to a Lorentz transformation, of this form. In particular,* l, l_1, l_2, l_3 *must be linearly dependent.*

Proof: The first claim was already proven in 3.5.2.5. Choose an arbitrary $g \in dS^4$ orthogonal to l, l_1, l_2. It follows

$$\pm g \;\in\; \overline{W[l,l_1] \cap W[l,l_2]} \;\subset\; \overline{W[l,l_3]},$$

i.e., $\pm g = \lambda_{\pm} l - \mu_{\pm} l_3 + k_{\pm}$ for $\lambda_{\pm}, \mu_{\pm} \geq 0$ and $k_{\pm} \in \mathbf{R}^5$ satisfying $(k_{\pm}, l) = 0 = (k_{\pm}, l_3)$. This shows

$$0 \;=\; g - g \;=\; \lambda_+ l - \mu_+ l_3 + k_+ + \lambda_- l - \mu_- l_3 + k_-$$

implying $\lambda_{\pm} = \mu_{\pm} = 0$, resp. $g \in \{l, l_1, l_2, l_3\}^{\perp}$, i.e., l_3 is a linear combination of l, l_1, l_2. It remains to show which a_1, a_2, b_1, b_2 satisfy (4.1.2) (we can choose the coordinate system suitably). Since $W_R \cap W[l_{1+}, (1,a,b,0,0)^{\mathrm{T}}]$, $b \geq 0$, is strictly decreasing in a (the proof of Lemma 3.5.2.5 implies that this intersection would otherwise be independent of a), it suffices to prove

$$W_R \cap W[l_{1+}, (1,a',b',0,0)^{\mathrm{T}}] \;\not\subset\; W[l_{1+}, (1,a,b,0,0)^{\mathrm{T}}],$$

for $a^2 + b^2 = 1 = a'^2 + b'^2$, $b'b < 0$, resp.

$$W_R \cap W[l_{1+}, l_{2\pm}] \;\not\subset\; W[l_{1+}, (1,a,b,0,0)^{\mathrm{T}}],$$

for $\pm b < 0$ (transform the coordinate system with a boost in the direction of the x_1-axis). It follows easily

$$(0,0,\mp 1,0,0)^{\mathrm{T}} = l_{1+} - (1,1,\pm 1,0)^{\mathrm{T}} \in \overline{W_R \cap W[l_{1+}, l_{2\pm}]}.$$

However,

$$(0,0,\mp 1,0,0)^{\mathrm{T}} = \pm\frac{b}{1-a} l_{1+} \mp \frac{1}{1-a}(b,b,1-a,0,0)^{\mathrm{T}} \notin \overline{W[l_{1+}, (1,a,b,0,0)^{\mathrm{T}}]}$$

$(\pm\frac{b}{1-a} < 0)$.

Lemma 4.1.5: *To any future-directed lightlike l there are future-directed lightlike l_1, l_2, l_3, h, h_1, h_2, h_3 such that l, l_1, l_2, l_3 as well as h, h_1, h_2, h_3 are linearly independent and $\tau W[l, l_k] = W[h_k, h]$ for $k = 1, 2, 3$ or $\tau W[l, l_k] = W[h, h_k]$ for $k = 1, 2, 3$.*

Proof: According to Lemma 4.1.3, there are lightlike l'_1, l'_2, l'_3, l'_4 in the forward lightcone such that any four of the vectors $l, l'_1, l'_2, l'_3, l'_4$ are linearly independent and

$$\cap_{k=1}^{4} \tau W[l, l'_k] \neq \emptyset.$$

Hence, the pairs $(\tau W[l, l'_i], \tau W[l, l'_j])$, $i, j = 1, \ldots, 4$, $i \neq j$, are self-dual and we can easily conclude either $\tau W[l, l'_k] = W[h'_k, h]$ for $k = 1, 2, 3, 4$ or $\tau W[l, l'_k] = W[h, h'_k]$ for $k = 1, 2, 3, 4$, where $h, h'_1, h'_2, h'_3, h'_4$ are suitable future-directed lightlike vectors. We only consider the latter case, since the other case is similar. It is sufficient to show that $\mathrm{span}\{h, h'_1, h'_2, h'_3, h'_4\}$ is at least four-dimensional (then we can pick suitable $\{l_1, l_2, l_3\} \subset \{l'_1, l'_2, l'_3, l'_4\}$). Therefore, we derive a contradiction from the assumption that $h, h'_1, h'_2, h'_3, h'_4$ span a three-dimensional subspace, resp. that, for a suitable choice of the coordinate system, $W[h, h'_1] = W_R$ and h'_2, h'_3, h'_4 have trivial x_3- and x_4-components. Now it follows easily from (4.1.2) that there are $W_1, W_2 \in \{W[h, h'_2], W[h, h'_3], W[h, h'_4]\}$ such that $W_1 \neq W_2$ and

$$W[h, h'_1] \cap W_1 \; = \; W_R \cap W_1 \; \subset \; W_2,$$

since there are two vectors among h'_2, h'_3, h'_4 both of which have either positive or negative x_2-components. However, according to (4.1.1), this would imply a corresponding relation for the wedges $W[l, l'_k]$, $k = 1, 2, 3, 4$, in contradiction to the last statement in Lemma 4.1.4 and the fact that any four of the vectors $l, l'_1, l'_2, l'_3, l'_4$ are linearly independent.

Lemma 4.1.6: *For a given future-directed lighlike vector l, choose $l_1, l_2, l_3, h,\ h_1, h_2, h_3$ as in Lemma 4.1.5. Then, for any lightlike future-directed k, there are lightlike future-directed l', k' such that*

$$\tau W[l,k] \;=\; W[l',k'],$$

where a multiple of l' or of k' is equal to h.

Proof: Assume that neither l' nor k' is a multiple of h. Then, l', k', h are linearly independent, i.e., even l', k', h, h_j are linearly independent, for a suitable j, implying $W[l',k'] \cap W[h,h_j] \neq \emptyset$ and $W[l',k'] \cap W[h_j,h] \neq \emptyset$ (Lemma 4.1.1). Hence, the pair $(\tau W[l,k], \tau W[l,l_j])$, which is equal to $(W[l',k'], W[h_j,h])$ or $(W[l',k'], W[h,h_j])$, is self-dual. However, the fact that l', k', h, h_j are linearly independent contradicts the self-duality (Lemma 4.1.4).

Lemma 4.1.7: *If k, r, l are future-directed lightlike vectors which are pairwise linearly independent, then*

$$\tau W[k,r] \cap \tau W[k,l] \;\neq\; \emptyset. \tag{4.1.3}$$

Therefore, self-dual pairs of wedges are mapped under τ onto self-dual pairs.

Proof: Lemma 4.1.6 shows that to any future-directed lightlike l there are future-directed lightlike vectors $\tau_1 l, \tau_2 l$, uniquely determined up to multiples, such that

$$\{\tau W[l,k] \,|\, k \text{ future-directed lightlike}\}$$
$$\subset \{W[\tau_1 l, k], W[k, \tau_1 l] \,|\, k \text{ future-directed lightlike}\}$$

and

$$\{\tau W[k,l] \,|\, k \text{ future-directed lightlike}\}$$

$$\subset \{W[\tau_2 l, k], W[k, \tau_2 l] \mid k \text{ future-directed lightlike}\}.$$

The surjectivity of τ implies that any lightlike vector is, up to a multiple, an image under τ_1, resp. under τ_2.

There are pairwise linearly independent $l^{(t)}$, $t \geq 0$, such that

$$\emptyset \neq W[k,r] \cap W[k,l^{(t)}] \subset W[k,r] \cap W[k,l] \tag{4.1.4}$$

(Lemma 4.1.4).

Now we want to derive (4.1.3). Let k', r' be future-directed lightlike vectors such that $\tau W[k,r] = W[k',r']$. Choose a future-directed lightlike s such that $\tau_1 s$ is neither a multiple of k' nor of r'. Since $\{\tau W[s, l^{(t)}] \mid t \geq 0\}$ is not a finite set, i.e., since

$$\{\tau W[s, l^{(t)}] \mid t \geq 0\} \not\subset \{W[\tau_1 s, k'], W[k', \tau_1 s], W[\tau_1 s, r'], W[r', \tau_1 s]\},$$

we can conclude that there is a $t_0 \geq 0$ such that $\tau_2 l^{(t_0)}$ is neither a multiple of k' nor of r'. According to (4.1.1) and (4.1.4), it suffices to show

$$\tau W[k,r] \cap \tau W[k, l^{(t)}] \neq \emptyset,$$

for a $t \geq 0$, i.e., in order to prove (4.1.3), we can assume $l = l^{(t_0)}$, resp. that $\tau_2 l$ is neither a multiple of k' nor of r'.

Lemma 4.1.5 still holds if we replace $W[l, l_k]$ by $W[l_k, l]$, for $k = 1,2,3$. Furthermore, we can choose l_1, l_2, l_3 such that $l_1, l_2, l_3 \notin \operatorname{span}\{k, r, l\}$ (according to its proof, Lemma 4.1.3 holds for suitable $l_1, l_2, l_3, l_4 \notin \operatorname{span}\{k, r, l\}$). Note $\tau_2 l = h$ (Lemma 4.1.5). We can conclude

$$\emptyset \neq W[k,r] \cap W[l_i, l] \subset W[k,r] \cap W[k,l],$$

for $i = 1,2,3$, since k, r, l, l_i are linearly independent (Lemma 4.1.1).

There is a suitable i_0 such that $k', r', \tau_2 l = h, h_{i_0}$ are linearly independent, since we assumed that $\tau_2 l = h$ is neither a multiple of k' nor of r' (implying that $\operatorname{span}\{h, k', r'\}$ is three-dimensional) and since h, h_1, h_2, h_3, which are chosen as in Lemma 4.1.5, are linearly independent, i.e.,

$$\emptyset \neq W[k', r'] \cap W[h_{i_0}, h] \quad \text{and} \quad \emptyset \neq W[k', r'] \cap W[h, h_{i_0}]$$

(Lemma 4.1.1), resp.

$$\emptyset \neq \tau W[k, r] \cap \tau W[l_{i_0}, l]$$

(Lemma 4.1.5). (4.1.1) implies

$$\emptyset \neq \tau W[k, r] \cap \tau W[l_{i_0}, l] \subset \tau W[k, r] \cap \tau W[k, l].$$

The final statement of the lemma is a consequence of (4.1.1).

Lemma 4.1.8: *τ maps sets of the form $\{W[l, k] \mid l$ future-directed lightlike$\}$ and $\{W[k, l] \mid l$ future-directed lightlike$\}$ onto sets of the same form. Furthermore,*

$$\tau(W') = (\tau W)', \quad \text{for} \quad W \in \mathcal{W},$$

and either

$$\{\tau W[k, l] \mid l\} = \{W[k', l] \mid l\} \quad \textit{and} \quad \{\tau W[l, k] \mid l\} = \{W[l, k'] \mid l\},$$

for all future-directed lightlike k and suitable k', or

$$\{\tau W[k, l] \mid l\} = \{W[l, k'] \mid l\} \quad \textit{and} \quad \{\tau W[l, k] \mid l\} = \{W[k', l] \mid l\},$$

for all future-directed lightlike k and suitable k'.

Proof: If we consider τ and τ^{-1}, then the first claim follows from Lemma 4.1.7. $\mathcal{F}_1 = \{W[l, k] \mid l\}$ and $\mathcal{F}_2 = \{W[k, l] \mid l\}$ have trivial intersection and no pair (W_1, W_2),

$W_1 \in \mathcal{F}_1$, $W_2 \in \mathcal{F}_2$, is self-dual. $\mathcal{F}_2$ is the only family of the form $\{W[l,h]\,|\,l\}$ or $\{W[h,l]\,|\,l\}$ such that this property is satisfied. This relation must also hold for $\tau\mathcal{F}_1 = \{\tau W[l,k]\,|\,l\}$ and $\tau\mathcal{F}_2 = \{\tau W[k,l]\,|\,l\}$, i.e., to every future-directed lightlike k there is a future-directed lightlike k' such that either

$$\tau\mathcal{F}_1 \;=\; \{W[l,k']\,|\,l\} \quad\text{and}\quad \tau\mathcal{F}_2 \;=\; \{W[k',l]\,|\,l\}$$

or

$$\tau\mathcal{F}_1 \;=\; \{W[k',l]\,|\,l\} \quad\text{and}\quad \tau\mathcal{F}_2 \;=\; \{W[l,k']\,|\,l\}.$$

Let h be a future-directed lightlike vector such that h, k are linearly independent. $\mathcal{F}_1 = \{W[l,k]\,|\,l\}$ and $\{W[h,l]\,|\,l\}$ have exactly one element in common ($W[h,k]$). Similar relations must hold for the images of these families of wedges under τ, i.e., either

$$\{\tau W[l,h]\,|\,l\} \;=\; \{W[l,h']\,|\,l\} \quad\text{and}\quad \{\tau W[h,l]\,|\,l\} \;=\; \{W[h',l]\,|\,l\},$$

for a suitable future-directed lightlike h', if

$$\{\tau W[l,k]\,|\,l\} \;=\; \{W[l,k']\,|\,l\} \quad\text{and}\quad \{\tau W[k,l]\,|\,l\} \;=\; \{W[k',l]\,|\,l\}$$

or

$$\{\tau W[l,h]\,|\,l\} \;=\; \{W[h',l]\,|\,l\} \quad\text{and}\quad \{\tau W[h,l]\,|\,l\} \;=\; \{W[l,h']\,|\,l\},$$

for a suitable future-directed lightlike h', if

$$\{\tau W[l,k]\,|\,l\} \;=\; \{W[k',l]\,|\,l\} \quad\text{and}\quad \{\tau W[k,l]\,|\,l\} \;=\; \{W[l,k']\,|\,l\}.$$

Lemma 4.1.9:

$$\tau W_1 \cap \tau W_2 \;=\; \emptyset \;\Leftrightarrow\; W_1 \cap W_2 \;=\; \emptyset,$$

for $W_1, W_2 \in \mathcal{W}$.

Proof: It is sufficient to show that $W_1 \cap W_2 = \emptyset$ implies $\tau W_1 \cap \tau W_2 = \emptyset$ (consider τ and τ^{-1}). There are future-directed lightlike k_1, k_2, l_1, l_2 such that $W_1 = W[k_1, k_2]$ and $W_2 = W[l_1, l_2]$. Lemma 4.1.8 entails the claim if k_1, resp. k_2, is a multiple of l_2, resp. of l_1. Hence, we can assume that l_1, l_2, k_1, k_2 are pairwise linearly independent.

First we show, for pairwise linearly independent k_1, k_2, l_1, l_2,

$$W[k_1, k_2] \cap W[l_1, l_2] = \emptyset \;\Leftrightarrow\; W[l_1, k_2] \cap W[l_1, l_2] \subset W[l_1, k_1]. \tag{4.1.5}$$

Since we can restrict ourselves to the case of linearly dependent k_1, k_2, l_1, l_2 (Lemma 4.1.1 and Lemma 4.1.4), we can assume that $l_1 = l_{1+}, k_2 = l_{1-}, k_1 = l_{2+}$ and $l_2 = (1, a, b, 0, 0)^{\mathrm{T}}$ for $a^2 + b^2 = 1$.

$$\begin{aligned} W[k_1, k_2] \cap W[l_1, l_2] &= W[l_1, k_2] \cap W[k_1, l_2] \\ &= W_R \cap W[l_{2+}, (1, a, b, 0, 0)^{\mathrm{T}}] \\ &= \emptyset \end{aligned}$$

is equivalent to $a > 0, b > 0$ (recall that the vectors are pairwise linearly independent and use Lemma 4.1.1), which is equivalent to

$$W[l_1, k_2] \cap W[l_1, l_2] = W_R \cap W[l_{1+}, (1, a, b, 0, 0)^{\mathrm{T}}] \subset W[l_{1+}, l_{2+}] = W[l_1, k_1]$$

(Lemma 4.1.4).

Let l_1', l_2', k_1', k_2' be the lightlike vectors associated to l_1, l_2, k_1, k_2 by Lemma 4.1.8. By assuming $W_1 \cap W_2 = \emptyset$ and by applying τ to the preceding inclusion we can conclude

$$W[l_1', k_2'] \cap W[l_1', l_2'] \subset W[l_1', k_1']$$

or

$$W[k_2', l_1'] \cap W[l_2', l_1'] \subset W[k_1', l_1'].$$

In the first case, (4.1.5) implies

$$\tau W_1 \cap \tau W_2 \;=\; W[k_1', k_2'] \cap W[l_1', l_2'] \;=\; \emptyset.$$

The other case can be treated similarly.

Lemma 4.1.10: *τ induces a bijection of the characteristic planes, which we will call τ too, such that, for linearly independent lightlike l_1, l_2 in the forward lightcone, $\tau H_0[l_1]$ and $\tau H_0[l_2]$ are the characteristic planes determining $\tau W[l_1, l_2]$.*

Proof: Let l be an arbitrary lightlike vector in the forward lightcone. Then

$$\{\tau W[l,k]\,|\,k\} \;=\; \{W[l',k]\,|\,k\} \quad \text{or} \quad \{\tau W[l,k]\,|\,k\} \;=\; \{W[k,l']\,|\,k\},$$

for a suitable l'. Set $\tau H_0[l] = H_0[l']$. Lemma 4.1.8 entails the claim.

We can prove a lemma corresponding to Lemma 3.1.9 (consider pairs of wedges with trivial intersection instead of maximal pairs).

Lemma 4.1.11: *Let l_1, l_2, l_3, l_4 be lightlike vectors which are linearly dependent but pairwise linearly independent. Then*

$$\cap_{i=1}^{4} \tau H_0[l_i] \;=\; \cap_{i \neq k} \tau H_0[l_i], \quad k = 1, 2, 3, 4.$$

Corollary 4.1.12: *Let l_1, l_2, l_3, l_4, l_5 be nontrivial lightlike vectors with four-dimensional*

$$\text{span}\{l_1, \ldots, l_5\} \;=\; \text{span}\{l_1, \ldots, l_4\}.$$

Then

$$\cap_{i=1}^{5} \tau H_0[l_i] = \cap_{i=1}^{4} \tau H_0[l_i].$$

Proof: We can assume $l_5 \notin \operatorname{span}\{l_1, \ldots, l_3\}$. l_5 is a multiple of l_4 or it is a linear combination of l_4 and two lightlike vectors k_1, k_2 in $\operatorname{span}\{l_1, \ldots, l_3\}$.

$$\cap_{i=1}^{3} \tau H_0[l_i] = (\cap_{i=1}^{3} \tau H_0[l_i]) \cap \tau H_0[k_j], \quad j = 1, 2,$$

entails

$$\begin{aligned} \cap_{i=1}^{5} \tau H_0[l_i] &= (\cap_{i=1}^{3} \tau H_0[l_i]) \cap (\tau H_0[k_1] \cap \tau H_0[k_2] \cap \tau H_0[l_4] \cap \tau H_0[l_5]) \\ &= (\cap_{i=1}^{3} \tau H_0[l_i]) \cap (\tau H_0[k_1] \cap \tau H_0[k_2] \cap \tau H_0[l_4]) \\ &= \cap_{i=1}^{4} \tau H_0[l_i]). \end{aligned}$$

Corollary 4.1.13: *Let $x \in \mathbf{R}^5$ be spacelike. Then*

$$\cap_{x \in H_0[l]} \tau H_0[l]$$

is one-dimensional, i.e., τ induces a bijection of the one-dimensional subspaces of $\mathbf{R}^5$ which consist of spacelike vectors. We will call this map τ too.

Proof: Let l_1, l_2, l_3, l_4 be linearly independent lightlike vectors such that $x \in H_0[l_i]$, for $i = 1, 2, 3, 4$. Then

$$\cap_{x \in H_0[l]} \tau H_0[l] = \cap_{i=1}^{4} \tau H_0[l_i],$$

according to Corollary 4.1.12 (if x is contained in five characteristic planes, then the five corresponding lightlike vectors, which are determined up to multiples, are linearly dependent).

Lemma 4.1.14: *Let $x \in \mathbf{R}^5$ be spacelike and $a_k, b_k, c_k, d_k \in \mathbf{R}$ ($k = 1, 2$, $a_k^2 + b_k^2 + c_k^2 + d_k^2 = 1$). Set*

$$W_1 = W[(1, a_1, b_1, c_1, d_1)^{\mathrm{T}}, (1, a_2, b_2, c_2, d_2)^{\mathrm{T}}].$$

$\mathbf{R}x \cap W_1 \neq \emptyset$ is equivalent to $W_1 \cap W_2 \neq \emptyset$ for all $W_2 \in \mathcal{W}$ whose edge contains x. For $x = (0,0,0,0,1)^{\mathrm{T}}$ this is also equivalent to $d_1 d_2 < 0$. $(0,0,0,0,1)^{\mathrm{T}} \in W_1$ implies $d_1 > 0$ and $-(0,0,0,0,1)^{\mathrm{T}} \in W_1$ implies $d_1 < 0$ (if $d_1 d_2 < 0$).

Proof: We can assume $x = (0,0,0,0,1)^{\mathrm{T}}$. $\mathbf{R}x \cap W_1 \neq \emptyset$ implies $W_1 \cap W_2 \neq \emptyset$ for all $W_2 \in \mathcal{W}$ whose edge contains x (W_1 is open).

First we show that $W_1 \cap W_2 \neq \emptyset$ for all $W_2 \in \mathcal{W}$ whose edge contains x implies $d_1 d_2 < 0$. $d_1 = d_2 = 0$ is impossible for this would contradict $W_1 \cap W_1' = \emptyset$. Consider W_1' instead of W_1 if $d_1 = 0$. Therefore, we can assume $d_1 \neq 0$. By applying suitable Lorentz transformations, which leave $(0,0,0,0,1)^{\mathrm{T}}$ invariant, we can further assume $a_1 = b_1 = c_1 = 0$ and, by applying a suitable rotation, $b_2 = c_2 = 0$. If $d_1 = 1$, $d_2 \geq 0$, and $a_2 \geq 0$, we know $W_1 \cap W_R = \emptyset$ and if $d_1 = 1$, $d_2 \geq 0$, and $a_2 \leq 0$, then $W_1 \cap W_R' = \emptyset$ (Lemma 4.1.1). We can also find wedges which contain $(0,0,0,0,1)^{\mathrm{T}}$ in their edge and possess trivial intersection with W_1 if $d_1 = -1$ and $d_2 \leq 0$ (Lemma 4.1.1). Altogether we have shown that $W_1 \cap W_2 \neq \emptyset$ for all $W_2 \in \mathcal{W}$ whose edge contains x implies $d_1 d_2 < 0$.

Suppose now $d_1 d_2 < 0$. Again we can assume $a_1 = b_1 = c_1 = b_2 = c_2 = 0$. The claim follows for $d_1 = 1$, $d_2 < 0$ (a similar formula is also valid for $d_1 = -1$) with

$$\begin{aligned} &(0,0,0,0,(1-d_2)^2)^{\mathrm{T}} \\ = \; &-d_2(1-d_2)(1,0,0,0,1)^{\mathrm{T}} - (1-d_2)(1,a_2,0,0,d_2)^{\mathrm{T}} \\ &+a_2(a_2, 1-d_2, 0, 0, a_2)^{\mathrm{T}} \end{aligned}$$

$$\in (1-d_2)^2 W_1.$$

Lemma 4.1.15: *Let $x \in \mathbf{R}^5$ be spacelike. If $x \in W_1 \cap W_2$ for $W_1, W_2 \in \mathcal{W}$, then*

$$\emptyset \neq \tau \mathbf{R}x \cap \tau W_1 = \tau \mathbf{R}x \cap \tau W_2,$$

i.e. we can implement τ by a bijection of de Sitter space.

Proof: τ maps the set of wedges whose edges contain $\mathbf{R}x$ onto the set of wedges whose edges contain $\tau\mathbf{R}x$. $\tau\mathbf{R}x \cap \tau W_1 \neq \emptyset$ and $\tau\mathbf{R}x \cap \tau W_2 \neq \emptyset$ are a consequence of Lemma 4.1.14. There are lightlike vectors l_1, l_2, l_3, l_4 such that $\tau W_1 = W[l_1, l_2]$ and $\tau W_2 = W[l_3, l_4]$. Lemma 4.1.8 and Lemma 4.1.14 imply $\mathbf{R}x \cap \tau^{-1} W[l_1, l_4] \neq \emptyset$ and therefore $\tau\mathbf{R}x \cap W[l_1, l_4] \neq \emptyset$. Lemma 4.1.14 entails now

$$\tau\mathbf{R}x \cap W[l_1, l_2] = \tau\mathbf{R}x \cap W[l_1, l_4] = \tau\mathbf{R}x \cap W[l_3, l_4].$$

Now it follows easily that τ can be implemented by a Lorentz transformation (see [21] or the proof of [18, Theorem 6.1.10]).

4.2 Construction of the de Sitter Group

Under the assumption of the strong CGMA and that the modular conjugations act transitively on the net we will derive the existence of a strongly continuous representation of the proper de Sitter group.

In the following we will repeatedly use Lemma 3.2.2.1. Let $g \in \mathcal{P}$ be an involution with n-dimensional eigenspace corresponding to the eigenvalue -1. We will say that g is of type n if g does not have a timelike eigenvector corresponding to the eigenvalue -1. Otherwise, g is of type $-n$. First we want to show that the involutions g_W, $W \in \mathcal{W}$, are of type -2 and $\mathcal{S}_+^\uparrow \subset \mathcal{G}$. The latter relation implies

$$g_{W_R} \;=\; \mathrm{diag}(-1,-1,1,1,1) \tag{4.2.1}$$

(see the proof of [18, Proposition 4.2.10]). We can find the action g_W, $W \in \mathcal{W}$, of the other modular conjugations by adjoining g_{W_R} by an element of the de Sitter group which maps W_R onto W.

Note that the involutions g_W, $W \in \mathcal{W}$, are all of the same type (transitivity) and that even products of modular conjugations act as elements of $\mathcal{S}_+^\uparrow$ (see the proof of Theorem 3.3.1).

Lemma 4.2.1: *Assume that the net* $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ *satisfies the strong CGMA and that the modular conjugations* J_W, $W \in \mathcal{W}$, *act transitively on the net. If the involution* $\pm\mathrm{diag}(1,-1,-1,-1,-1)$ *is an element of* $\mathcal{G}$, *then* $\mathcal{G} \supset \mathcal{S}_+^\uparrow$.

Proof: Every element of $\mathcal{S}_+^\uparrow$ is of the form BR, where R is a product of rotations and where B is a boost [24]. Note

$$(\pm\mathrm{diag}(1,-1,-1,-1,-1))(BR)(\pm\mathrm{diag}(1,-1,-1,-1,-1))(BR)^{-1} \;=\; B^{-2}.$$

If $\mathcal{G}$ contains a boost, then we can prove the claim as Theorem 3.3.1. Hence, we can assume that $\mathcal{G} \cap \mathcal{S}_+^\uparrow$ only contains transformations which are products of rotations, i.e., $\mathcal{G}$ consists of elements of the form R and Rg_{W_R}, where R is a product of rotations. However, this contradicts the transitivity, since the rotation group possesses infinitely many orbits if we consider its action on $\mathcal{W}$.

Lemma 4.2.2: *Assume that the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ satisfies the strong CGMA and that the modular conjugations J_W, $W \in \mathcal{W}$, act transitively on the net. Let the wedge algebras $\mathcal{R}(W)$, $W \in \mathcal{W}$, be nonabelian. The involutions g_W, $W \in \mathcal{W}$, cannot be of type* 1, 2 *or* 3.

Proof: We show in each case $\mathcal{G} \supset \mathcal{S}_+^\uparrow$. Then it easily follows (4.2.1), resp. that the involutions g_W, $W \in \mathcal{W}$, are of type -2, in contradiction to the assumption (see the proof of [18, Proposition 4.2.10]).

Let the involutions g_W, $W \in \mathcal{W}$, be of type 1. By choosing the coordinate system suitably we can assume

$$g_W = \mathrm{diag}(1,1,1,1,-1),$$

for a $W \in \mathcal{W}$. g_W commutes with g_{W_R} implying that $g_W g_{W_R}$ is an involution of type 2 ($g_{W_R} g_W W_R = g_{W_R} W_R \neq W_R$ is a consequence of the nonabelianness of the wedge algebras). By choosing the coordinate system suitably we can assume $g = \mathrm{diag}(1,1,1,-1,-1) \in \mathcal{G}$. For this new choice of coordinate system, we can conclude that $g\, g_{W_R} \in \mathcal{G}$ is an involution of type 3, since $g\, g_{W_R}$ cannot be of type 1 ($g_{W_R} g W_R = g_{W_R} W_R \neq W_R$). Now the claim follows easily in this case ($g\, g_{W_R}$ lies in the center of the invariance group of a suitable wedge; see the proof of Theorem 3.3.1). This remark also excludes the occurence of type 3 involutions g_W.

It remains to consider the case of type 2 involutions g_W. We can assume $\mathrm{diag}(1,1,1,-1,-1) \in \mathcal{G}$. It commutes with g_{W_R}. If $g_{W_R}\mathrm{diag}(1,1,1,-1,-1)$ is of type 4, then Lemma 4.2.1 entails the claim. Hence, we can assume that the involution $g_{W_R}\mathrm{diag}(1,1,1,-1,-1)$ is of type 2 ($g_{W_R} = \mathrm{diag}(1,1,1,-1,-1)$ would contradict the nonabelianness of the wedge algebras). By choosing the coordinate system suitably, we can assume

$$\mathrm{diag}(1,1,1,-1,-1),\ \mathrm{diag}(1,1,-1,-1,1),\ \mathrm{diag}(1,1,-1,1,-1)\ \in\ \mathcal{G}$$

(g_{W_R}, $\mathrm{diag}(1,1,1,-1,-1)$, and $g_{W_R}\mathrm{diag}(1,1,1,-1,-1)$ are of type 2 and commute pairwise). For this new choice of coordinate system, we can conclude that g_{W_R} commutes with these involutions. The nonabelianness of the wedge algebras implies that the involutions

$$g_{W_R}\mathrm{diag}(1,1,1,-1,-1),\ g_{W_R}\mathrm{diag}(1,1,-1,-1,1),\ g_{W_R}\mathrm{diag}(1,1,-1,1,-1)$$

are nontrivial, i.e., one of them must be of type 4. The claim follows from Lemma 4.2.1.

Lemma 4.2.3: *Assume that the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ satisfies the strong CGMA and that the modular conjugations J_W, $W \in \mathcal{W}$, act transitively on the net. Let the wedge algebras $\mathcal{R}(W)$, $W \in \mathcal{W}$, be nonabelian. The involutions g_W, $W \in \mathcal{W}$, cannot be of type -1, -3 or -5.*

Proof: The transitivity excludes the last case. If the involutions g_W, $W \in \mathcal{W}$, would be of type -1, then we could assume $\mathrm{diag}(-1,1,1,1,1) \in \mathcal{G}$ and Lemma 4.2.1 would entail the claim.

It remains to consider the case of type -3 involutions. To any $W \in \mathcal{W}$, there is a $W_1 \in \mathcal{W}$ such that $g_W W_1 = W_1$. Hence $g_W g_{W_1}$ is an involution of type 2 or of type 4 ($g_{W_1} g_W W_1 \neq W_1$ implies $g_W g_{W_1} \neq 1$). It suffices to consider the first case (Lemma 4.2.1), i.e., we can assume $\mathrm{diag}(1,1,1,-1,-1) \in \mathcal{G}$.

In order to show that wedge duality is satisfied we merely have to consider the case of a type -3 involution $g_{W_R}\mathrm{diag}(1,1,1,-1,-1)$ (this involution is of type

-1, -3 or -5; the last case would imply wedge duality and the first one is excluded by Lemma 4.2.1). There is a $W \in \mathcal{W}$ such that

$$g_{W_R} W \;=\; \mathrm{diag}(1,1,1,-1,-1)W \;=\; W$$

(the restriction of g_{W_R} to the first three components x_0, x_1, and x_2 is of type -2, i.e., it has a spacelike eigenvector corresponding to the eigenvalue 1). Hence, g_W commutes with g_{W_R} and with $\mathrm{diag}(1,1,1,-1,-1)$. The nonabelianness of the wedge algebras implies either that $\mathcal{G}$ contains a type 4 involution ($g_W g_{W_R}$ or $g_W g_{W_R}\mathrm{diag}(1,1,1,-1,-1)$) or that $g_W g_{W_R}$ and $\mathrm{diag}(1,1,1,-1,-1)$ generate a group isomorphic to the Klein group such that its nontrivial elements are of type 2 (note that $g_W g_{W_R} \neq 1$ and $g_W g_{W_R} \neq \mathrm{diag}(1,1,1,-1,-1)$ follow from $\mathrm{diag}(1,1,1,-1,-1)W = W$ and $W \neq g_W W = g_W g_{W_R} W$). In the latter case we can find a $W_1 \in \mathcal{W}$ such that this group leaves W_1 invariant. The elements of this group commute with g_{W_1} implying $g_{W_1} = \mathrm{diag}(A, \pm 1, \pm 1, \pm 1)$ for a suitable 2×2-matrix A and a coordinate system which is chosen such that the elements of the Klein group are diagonal and leave $W_R = W_1$ invariant. We can conclude $g_{W_1} W_1 \in \{W_1, W_1'\}$, resp. $g_{W_1} W_1 = W_1'$, i.e., we can assume in any case that wedge duality is fulfilled.

We have to find a contradiction to the following assumptions: The modular conjugations act as involutions of type -3, wedge duality is satisfied, and $\mathrm{diag}(1,1,1,-1,-1) \in \mathcal{G}$. Wedge duality implies that this transformation commutes with $g_{W_4^{(0)}}$. According to Lemma 4.2.1 we can restrict ourselves to the case of a type -3 involution $g_{W_4^{(0)}}\mathrm{diag}(1,1,1,-1,-1)$ (the nonabelianness of the wedge algebras implies that this involution cannot be of type -5), i.e., we can assume $g_{W_4^{(0)}} = \mathrm{diag}(-1,-1,1,1,-1)$ (suitably rotate the coordinate system about the x_3x_4-plane).

As in the proof of wedge duality we can show either that $\mathcal{G}$ contains a type 4 involution or that $g_{W_2^{(0)}} g_{W_4^{(0)}}$ and $\mathrm{diag}(1,1,1,-1,-1)$ generate a subgroup of $\mathcal{G}$ which is isomorphic to the Klein group and whose nontrivial elements are of type 2. Thus, for a suitable choice of the coordinate system, we can assume that these transformations are of the form $\mathrm{diag}(1,1,\pm 1,\pm 1,\pm 1)$ and, in addition, $g_{W_R} = \mathrm{diag}(-1,-1,-1,1,1)$. $g_{W_4^{(0)}}$ commutes with the involutions g_{W_R}, $\mathrm{diag}(1,1,1,-1,-1)$, $\mathrm{diag}(1,1,-1,1,-1)$, and $\mathrm{diag}(1,1,-1,-1,1)$ which lie in $\mathcal{G}$, i.e., $g_{W_4^{(0)}} = \mathrm{diag}(A,\pm 1,\pm 1,\pm 1)$, where A is a suitable 2×2-matrix. Wedge duality implies $g_{W_4^{(0)}} = \mathrm{diag}(-1,\pm 1,\pm 1,\pm 1,-1)$. We can assume

$$g_{W_4^{(0)}} = \mathrm{diag}(-1,-1,1,1,-1), \tag{4.2.2}$$

since in the other cases we can show $\mathrm{diag}(-1,1,1,1,1) \in \mathcal{G}$ (note that $g_{W_4^{(0)}}$ is of type -3 and consider the product of $g_{W_4^{(0)}}$ and of a suitable type 2 involution $\mathrm{diag}(1,1,\pm 1,\pm 1,\pm 1) \in \mathcal{G}$). Furthermore, there is a group isomorphic to the Klein group which leaves $W_4^{(0)}$ invariant, i.e., whose elements commute with $g_{W_4^{(0)}}$. This group contains an element of the form

$$R\,\mathrm{diag}(1,-1,-1,1,1)R^{-1} \in \mathcal{G},$$

where R is a rotation about the x_1x_4-plane (recall (4.2.2)). Wedge duality implies that this transformation commutes with $g_{W_R} = \mathrm{diag}(-1,-1,-1,1,1)$. Hence, R is a rotation by a multiple of $\pi/2$. We can conclude

$$\mathrm{diag}(1,-1,-1,1,1) \in \mathcal{G} \quad \text{or} \quad \mathrm{diag}(1,-1,1,-1,1) \in \mathcal{G},$$

resp.

$$\mathrm{diag}(1,-1,-1,-1,-1) = \mathrm{diag}(1,-1,-1,1,1)\,\mathrm{diag}(1,1,1,-1,-1) \in \mathcal{G}$$

or

$$\mathrm{diag}(1,-1,-1,-1,-1) \;=\; \mathrm{diag}(1,-1,1,-1,1)\,\mathrm{diag}(1,1,-1,1,-1) \in \mathcal{G}.$$

The claim follows from Lemma 4.2.1.

Proposition 4.2.4: *Assume that the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ satisfies the strong CGMA and that the modular conjugations J_W, $W \in \mathcal{W}$, act transitively on the net. Let the wedge algebras $\mathcal{R}(W)$, $W \in \mathcal{W}$, be nonabelian. The involutions g_W, $W \in \mathcal{W}$, are of type -2 and $\mathcal{G} \supset \mathcal{S}_+^\uparrow$. J_{W_R} acts as $g_{W_R} = \mathrm{diag}(-1,-1,1,1,1)$.*

Proof: If $\mathcal{G}$ contains a type -2 involution, then we can show $\mathcal{G} \supset \mathcal{S}_+^\uparrow$ (see the proof of Theorem 3.3.1 and note that such an involution g lies in the centralizer of the invariance group of the wedge $W \in \mathcal{W}$ which satisfies $gW = W'$). Since $\mathcal{G} \supset \mathcal{S}_+^\uparrow$ implies (4.2.1) (see the proof of [18, Proposition 4.2.10]), it suffices to exclude that the involutions g_W are of type -4 (Lemma 4.2.1, Lemma 4.2.2, and Lemma 4.2.3). We can assume $\mathrm{diag}(-1,-1,-1,-1,1) \in \mathcal{G}$. Since the wedge algebras are nonabelian, $\mathrm{diag}(-1,-1,-1,-1,1)g_{W_4^{(0)}} \in \mathcal{G}$ is of type 2. We consider a type -4 involution $g = \mathrm{diag}(-1,-1,-1,-1,1) \in \mathcal{G}$ (choose the coordinate system suitably). Because of the transitivity and the existence of type 2 involutions in $\mathcal{G}$, we can conclude that $\mathcal{G}$ contains an involution h of type 2 which leaves $W_4^{(0)}$ invariant. $gh = hg$ is a type -2 involution, i.e., the claim follows.

Theorem 4.2.5: *Assume that the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ satisfies the strong CGMA and that the modular conjugations J_W, $W \in \mathcal{W}$, act transitively on the net. Let Ω be cyclic for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$ if $W_1, W_2 \in \mathcal{W}$, $W_1 \cap W_2 \neq \emptyset$. There is a strongly continuous representation U of the proper de Sitter group $\mathcal{S}_+$, which acts geometrically* *correctly on the net, such that* $U(\mathcal{S}_+) = \mathcal{J}$ *and* $U(g_W) = J_W$ *for* $W \in \mathcal{W}$*.*

Proof: The wedge algebras cannot be abelian, since otherwise $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$ would be maximally abelian for $W_1, W_2 \in \mathcal{W}$, $W_1 \cap W_2 \neq \emptyset$. The claim can be proven as Theorem 3.5.3.2.

REFERENCES

[1] A.D. Alexandrov, On Lorentz transformations, *Uspehi Mat. Nauk.*, **5**, 187 (1950).

[2] A.D. Alexandrov, Mappings of spaces with families of cones and space-time transformations, *Annali di Mat. Pura Appl.*, **103**, 229-257 (1975).

[3] H. Araki, A generalization of Borchers theorem, *Helv. Phys. Acta*, **36**, 132-139 (1963).

[4] J. Bisognano, E.H. Wichmann, On the duality condition for a hermitian scalar field, *J. Math. Phys.*, **16**, 985-1007 (1975).

[5] J. Bisognano, E.H. Wichmann, On the duality condition for quantum fields, *J. Math. Phys.*, **17**, 303-321 (1976).

[6] H.-J. Borchers, Über die Vollständigkeit lorentzinvarianter Felder in einer zeitartigen Röhre, *Nuovo Cimento*, **19**, 787-793 (1960).

[7] H.-J. Borchers, The CPT-theorem in two-dimensional theories of local observables, *Commun. Math. Phys.*, **143**, 315-332 (1992).

[8] H.-J. Borchers, Half-sided modular inclusion and the construction of the Poincaré group, *Commun. Math. Phys.*, **179**, 703-723 (1996).

[9] H.-J. Borchers, D. Buchholz, Global properties of vacuum states in de Sitter space, ESI-preprint and gr-qc/9803036.

[10] H.-J. Borchers, Half-sided translations and the type of von Neumann algebras, *Lett. Math. Phys.*, **44**, 283-290 (1998).

[11] O. Brattelli, D. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Berlin, Heidelberg, New York: Springer-Verlag, 1979.

[12] J. Bros, H. Epstein, U. Moschella, Analyticity properties and thermal effects for general quantum field theory on de Sitter space-time, *Commun. Math. Phys.*, **196**, 535-570 (1998).

[13] R. Brunetti, D. Guido, R. Longo, Group cohomology, modular theory and space-time symmetries, *Rev. Math. Phys.*, **7**, 57-71 (1995).

[14] D. Buchholz, Product states for local algebras, *Commun. Math. Phys.*, **36**, 287-304 (1974).

[15] D. Buchholz, On the structure of local quantum fields with non-trivial interactions, *Proceedings of the International Conference on Operator Algebras, Ideals and their Applications in Theoretical Physics*, 146-153 (1978).

[16] D. Buchholz, C. D'Antoni, R. Longo, Nuclear maps and modular structure I, *J. Funct. Anal.*, **88**, 233-250 (1990).

[17] D. Buchholz, S.J. Summers, An algebraic characterization of vacuum states in Minkowski space, *Commun. Math. Phys.*, **155**, 449-458 (1993).

[18] D. Buchholz, O. Dreyer, M. Florig, S.J. Summers, Geometric modular action and spacetime symmetry groups, *Rev. Math. Phys.*, to appear.

[19] D. Buchholz, M. Florig, S.J. Summers, An algebraic characterization of vacuum states in Minkowski space, II, preprint.

[20] S. Doplicher, An algebraic spectrum condition, *Commun. Math. Phys.*, **1**, 1-5 (1965).

[21] O. Dreyer, Das Prinzip der geometrischen Wirkung im de Sitter-Raum, Diplomarbeit, University of Hamburg, 1996.

[22] M. Florig, On Borchers' theorem, *Lett. Math. Phys.*, **46**, 289-293 (1998).

[23] K. Fredenhagen, M. Jörß, Conformal Haag-Kastler nets, pointlike localized fields and the existence of operator product expansion, *Commun. Math. Phys.*, **176**, 541-554 (1996).

[24] I.M. Gel'fand, R.A. Minlos, Z.Y. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications*, New York: Macmillan, 1963.

[25] D. Guido, R. Longo, An algebraic spin and statistics theorem I, *Commun. Math. Phys.*, **172**, 517-533 (1995).

[26] R. Haag, *Local Quantum Physics*, Berlin, Heidelberg, New York: Springer-Verlag, 1993.

[27] J. Hilgert, K.-H. Neeb, *Lie-Gruppen und Lie-Algebren*, Vieweg, 1991.

[28] P.D. Hislop, R. Longo, Modular structure of the local algebras associated with the free massless scalar field theory, *Commun. Math. Phys.*, **84**, 71-85 (1982).

[29] B.S. Kay, R.M. Wald, Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on space-times with a bifurcate Killing horizon, *Phys. Rep.*, **207**, 49-136 (1991).

[30] K. Kraus, An algebraic spectrum condition, *Commun. Math. Phys.*, **16**, 138-141 (1970).

[31] C.C. Moore, Extensions and low dimensional cohomology theory of locally compact groups, I, *Trans. Amer. Math. Soc.*, **113**, 40-63 (1964).

[32] C.C. Moore, Extensions and low dimensional cohomology theory of locally compact groups, II, *Trans. Amer. Math. Soc.*, **113**, 64-86 (1964).

[33] C.C. Moore, Group extensions and cohomology for locally compact groups, III, *Trans. Amer. Math. Soc.*, **221**, 1-33 (1976).

[34] C.C. Moore, Group extensions and cohomology for locally compact groups, IV, *Trans. Amer. Math. Soc.*, **221**, 35-58 (1976).

[35] M.-J. Radzikowski, The Hadamard condition and Kay's conjecture in (axiomatic) quantum field theory on curved space-times, Ph.D. Dissertation, Princeton University, 1992.

[36] S.J. Summers, Geometric modular action and transformation groups, *Ann. Inst. Henri Poincaré*, **64**, 409-432 (1996).

[37] S.J. Summers, R. Verch, Modular inclusion, the Hawking temperature, and quantum field theory in curved space-time, *Lett. Math. Phys.*, **37**, 145-158 (1996).

[38] M. Takesaki, *Theory of Operator Algebras I*, New York, Heidelberg, Berlin: Springer-Verlag, 1979.

[39] L.J. Thomas, E.H. Wichmann, Standard forms of local nets in quantum field theory, *J. Math. Phys.*, **39**, 2643-2681 (1998).

[40] H.-W. Wiesbrock, A comment on a recent work of Borchers, *Lett. Math. Phys.*, **25**, 157-159 (1992).

[41] H.-W. Wiesbrock, Half-sided modular inclusions of von Neumann algebras, *Commun. Math. Phys.*, **157**, 83-92 (1993).

[42] H.-W. Wiesbrock, Conformal quantum field theory and half-sided modular inclusions of von Neumann algebras, *Commun. Math. Phys.*, **158** , 537-543 (1993).

[43] H.-W. Wiesbrock, Half-sided modular inclusions of von Neumann algebras, *Commun. Math. Phys.*, **184**, 683-685 (1997).

[44] H.-W. Wiesbrock, Modular intersections of von Neumann algebras in quantum field theory, *Commun. Math. Phys.*, **193**, 269-285 (1998).

[45] E.C. Zeeman, Causality implies the Lorentz group, *J. Math. Phys.*, **5**, 490-493 (1964).

BIOGRAPHICAL SKETCH

Martin Florig was born on July 25, 1967, in Aschaffenburg, Germany. From October 1986 to June 1995 he studied at the Julius-Maximilians-Universität in Würzburg. In January 1993 he received his diploma in mathematics and in June 1995 his diploma in physics. He received a Ph.D. in mathematics from the University of Florida in 1999.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Stephen J. Summers

Stephen J. Summers, Chairman
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Paul Ehrlich

Paul Ehrlich
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

John R. Klauder

John Klauder
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

P C R

Paul Robinson
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

K. C. Muttalib

Khandker Muttalib
Associate Professor of Physics

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

May 1999

Dean, Graduate School